No Practice Quiz, BUT a Quiz bye-bye shot, due at the end of the semester, that you can take at any time, up to May 13th.

Today: Infinite Limits

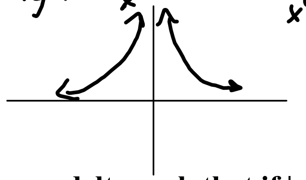
Algebra Review: Completing the Square to obtain a quick sketch of quadratic functions.

They come up a lot in math and on Quiz 1. You want to be able to whip these out, quickly.

Also a big application of the Squeeze Theorem.

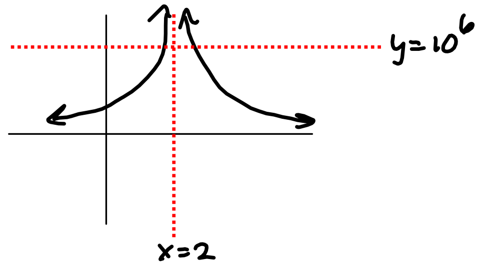
Infinite limits, more formally: $\frac{1}{x^2}$ $\qquad \frac{1}{x^4}, \frac{1}{x^6}, \frac{1}{x^8}, \ldots$

$\lim_{x \to 2} \frac{5}{(x-2)^2} = \infty$.



To prove that this limit is true, you want to show a delta such that if | x - 2 | < delta, then we can make the 5/(x-3)^2 bigger than any number you give us.

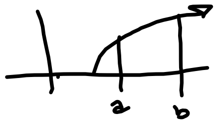
For a specific value of M: M = 1,000,000:



want $\frac{1}{(x-2)^2} > 1000000$

$\Longrightarrow \frac{1}{(x-2)^2} > 1000000$

$\Longrightarrow \frac{1}{1000000} > (x-2)^2$



$\Longrightarrow (x-2)^2 < \frac{1}{1000000}$

$\Longrightarrow \sqrt{(x-2)^2} < \sqrt{\frac{1}{1000000}}$ $\quad = \frac{\sqrt{1}}{\sqrt{1000000}} = \frac{1}{\sqrt{10^6}}$

$\Longrightarrow |x-2| < \frac{1}{10^3}$

Make $\delta = \frac{1}{10^3}$. Then $\frac{1}{(x-2)^2} > 1000000 = 10^6$

whenever $0 < |x-2| < \delta$.

In general:

Want $\frac{1}{(x-2)^2} > M$ for some (big) $M > 0$.

$\Longrightarrow \frac{1}{M} > (x-2)^2$,

i.e., $(x-2)^2 < \frac{1}{M}$

$\Longrightarrow \sqrt{(x-2)^2} < \sqrt{\frac{1}{M}} = \frac{\sqrt{1}}{\sqrt{M}} = \frac{1}{\sqrt{M}}$

$|x-2| < \frac{1}{\sqrt{M}}$.

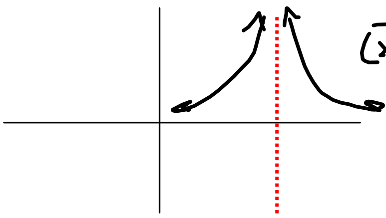
Let $\delta = \frac{1}{\sqrt{M}}$.

Quiz hint: Same thing works for any even power of (x - 2).

$\lim_{x \to 5} \frac{1}{(x-5)^4} = \infty$ goes the same way.

You just get $\sqrt[4]{\ }$ instead of $\sqrt{\ }$

$\frac{1}{(x-5)^4}$



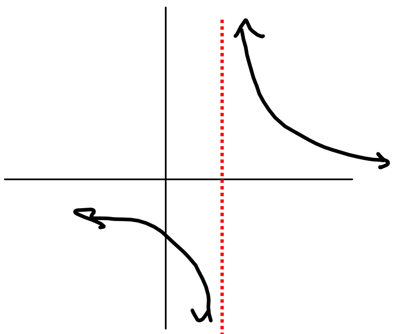
$x=5$

If the power is ODD, then one side of the limit will be +infinity and the other side of the limit, i.e., the "other-sided limit," will be -infinity.

$\lim_{x \to 2} \frac{1}{(x-2)^3} \nexists$

$\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$

$\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$



$x = 2$

When $x \to 2^-$, $x-2$ & $(x-2)^3$ & $(x-2)^5 \ldots$ are negative

When $x \to 2^+$ $x-2, (x-2)^3$, etc are positive.

$f(x) = \dfrac{1}{(x+3)^7}$ "blows up" @ $x=-3$

$\lim_{x\to -3^-} f(x) = -\infty$ &

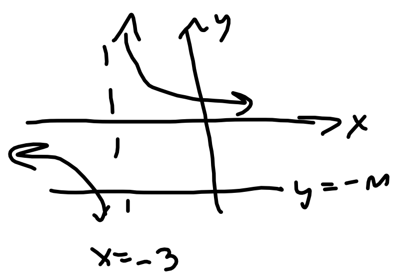
$\lim_{x\to -3^+} f(x) = +\infty.$

From the left, $\dfrac{1}{(x+3)^7} < 0$, i.e., is negative. (b/c $x+3<0$)

From the right, $\dfrac{1}{(x+3)^7} > 0$

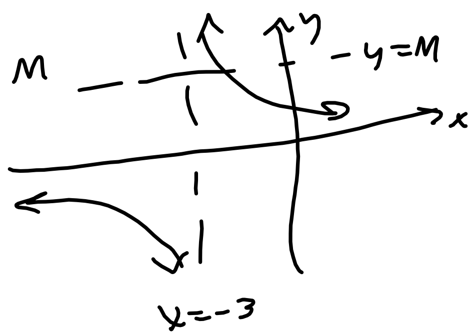
From the left, we want

want $\dfrac{1}{(x+3)^7} < -M$



From the right:

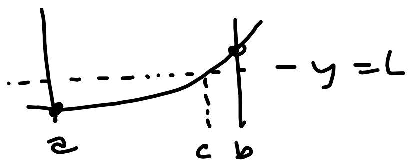
want $\dfrac{1}{(x+3)^7} > +M$



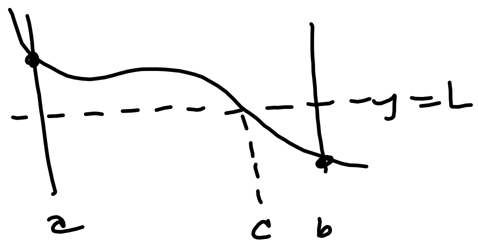
Intermediate Value Theorem.

If f is cnt^{s} on $[a,b]$, with $f(a) < f(b)$, with $f(a) < L < f(b)$, then there is a $c \in (a,b) \ni f(c) = L$.

If the chicken crosses the road, then it must cross the center line of the road.



Same thing for $f(a) > L > f(b)$



Is there a # that is exactly 3 more than its cube?

$$x = x^3 + 3 ?$$

$$\Rightarrow x^3 + x - 3 = 0$$

Let $f(x) = x^3 + x - 3$.

$f(-10) = -1000 - 10 - 3 < 0$

$f(10) = 1000 + 10 - 3 > 0$

$\therefore \exists\ c \in (-10, 10) \ni f(x) = 0$

This is equivalent to saying

$x^3 - x + 3 = 0$ has a root

This is equivalent to saying

$x^3 + 3 = x$, which was the original question.

Quadratic Functions.

Solving an equation

To solve:

$x^2 + 4x - 7 = 0$

$x^2 + 4x = 7$

$x^2 + 4x + 2^2 = 7 + 4$

$(x+2)^2 = 11$

$x + 2 = \pm\sqrt{11}$

$x = -2 \pm \sqrt{11}$

$\left(\frac{4}{2}\right)^2 = 2^2 = 4$

Manipulating an expression.

To "see"

$f(x) = x^2 + 4x - 7$

$= x^2 + 4x + 2^2 - 4 - 7$

$= (x+2)^2 - 11$

$(h, k) = (-2, -11)$

$(-2,-11)$

from $f(x) = a(x-h)^2 + k$

$2x^2+3x-7=0$ SEE $f(x)$ below:

$x^2+\frac{3}{2}x-\frac{7}{2}=0$

$x^2+\frac{3}{2}x \quad = \frac{7}{2}$ $\qquad \frac{\frac{3}{2}}{2} = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4} \rightsquigarrow \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

$x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2 = \frac{7}{2} \quad +\frac{9}{16}$

$\boxed{\left(x+\frac{3}{4}\right)^2} = \frac{7}{2}\cdot\frac{8}{8}+\frac{9}{16} = \frac{56+9}{16} \boxed{= \frac{65}{16}}$

$x+\frac{3}{4} = \pm\sqrt{\frac{65}{16}} = \pm\frac{\sqrt{65}}{\sqrt{16}} = \pm\frac{\sqrt{65}}{4}$

$x = -\frac{3}{4}\pm\frac{\sqrt{65}}{4}$

$f(x) = 2x^2+3x-7 \quad = 2\left(x^2+\frac{3}{2}x-\frac{7}{2}\right) \Longrightarrow$

$\Rightarrow \frac{1}{2}f(x) = x^2+\frac{3}{2}x-\frac{7}{2}$ $\qquad \left(\frac{\frac{3}{2}}{2}\right)^2 = \left(\frac{3}{2}\cdot\frac{1}{2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$= x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2-\frac{9}{16}-\frac{7}{2}\cdot\frac{8}{8}$

$\int \frac{\text{stuff}}{\sqrt{x^2+3x+1}}\,dx = \int \frac{\text{stuff}}{\sqrt{(x+\frac{3}{2})^2-\frac{5}{4}}}\,dx$

$x^2+3x+\left(\frac{3}{2}\right)^2-\frac{9}{4}+1 \qquad \sqrt{u^2-a^2}$

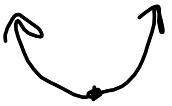
$= \left(x+\frac{3}{2}\right)^2-\frac{5}{4}$

$= \left(x+\frac{3}{4}\right)^2-\frac{9+56}{16}$

$= \left(x+\frac{3}{4}\right)^2-\frac{65}{16} = \frac{1}{2}f(x)$ (from above), so

$f(x) = 2\left[\left(x+\frac{3}{4}\right)^2-\frac{65}{16}\right] = 2\left(x+\frac{3}{4}\right)^2-2\left(\frac{65}{16}\right) = \boxed{2\left(x+\frac{3}{4}\right)^2-\frac{65}{8} = f(x)}$

$(h,k) = \left(-\frac{3}{4},-\frac{65}{8}\right)$



$\left(-\frac{3}{4},-\frac{65}{8}\right)$

Prove that

$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is cnt's everywhere.

Proof: $f(x)$ is cnt's everywhere except possibly $x=0$.

Squeeze Theorem

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

Assume $x > 0$. Then

$$-x \leq x\sin\frac{\pi}{x} \leq x$$

$x \to 0$

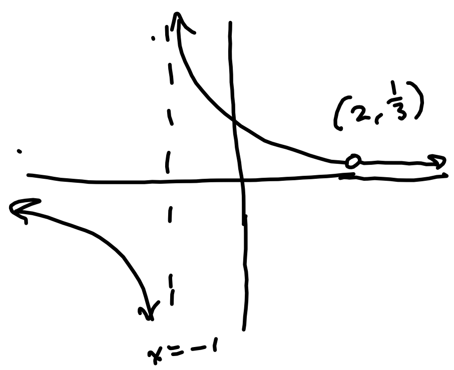
$$0 \leq \lim_{x \to 0^+} x\sin\left(\frac{\pi}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x \to 0^+} x\sin\left(\frac{\pi}{x}\right) = 0 \Rightarrow$$

$\lim_{x \to 0} f(x) = f(0)$, i.e., $x=0$ is a removable discontinuity.

$\frac{f(x+h)-f(x)}{h}$ always has a removable discontinuity @ $h=0$, if $f'(x)$ exists.

$$f(x) = \frac{x-2}{(x-2)(x+1)} = \frac{1}{x+1} \quad (x \neq 2)$$



Graph of $f(x)$

$x=2$ is a removable discontinuity.

Quiz 2 Notes that are inappropriate at this time.

Special: Variations on the "Topologist's Sine Curve"

$h(x) = \sin\left(\frac{\pi}{x}\right)$

$\sin(\theta) = 0$ when $\theta = n\pi,\ n \in \mathbb{Z}$ (i.e., $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$)

$\theta = \frac{\pi}{x} = n\pi \Rightarrow \frac{1}{x} = n \Rightarrow x = \frac{1}{n},\ \forall n \in \mathbb{Z} \setminus \{0\}$, i.e.,

$x = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots$

As we near $x=0$, $h(x)$ oscillates smoothly between $y = \pm 1$, but more and more rapidly.

Continuous and differentiable everywhere except x = 0.

Discontinuity can't be removed.

DAMPED SINE Curve

$f(x) = x\sin\left(\frac{\pi}{x}\right)$

Cnts and Difbl everywhere except x = 0

Has removable discontinuity at x = 0.

Removing the discontinuity does not make it differentiable.

"Double-damped" sine curve

$g(x) = x^2\sin\left(\frac{\pi}{x}\right)$

Cnts and Difbl everywhere except x = 0

Discontinuity at x = 0 is removable. When you remove the discontinuity you also get a differentiable function.