If you can get your Week 2 in before I post solutions, no deduction.

See Late Edition for Week 2.

It will close and I will open up a Late-Late Edition, with 30% discount that'll just stay open, indefinitely.

Let me know if your Week 1 didn't get graded. To see how I marked it up, go to Assignments, where you entered/uploaded your assignment. It's either graded, right there, OR there's an attachment with the annotations (grading).

Some of them were too big to leave in the same place and required an attachment, separately.

Section 1.7

The Precise Definition of a Limit

How close does x have to be to $x=3$ for $f(x)=3x-1$ to be less than 2 units from $y=8$?

The answer will be of the form $|x-3| <$ some #.

Note $f(3)=3(3)-1=8=$ functional value of f @ $x=3$.

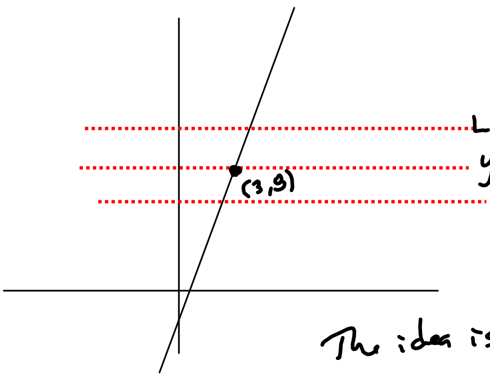
Want $|f(x)-8| < \boxed{2 \equiv \varepsilon}$

This means $|3x-1-8| = |3x-9| = |3(x-3)|$

$= |3||x-3| = 3|x-3| < 2 = \varepsilon$

$\Rightarrow |x-3| < \frac{2}{3} = \delta$

Define $\delta \equiv \frac{2}{3}$



$L+\varepsilon = L+2 = 10$
$y = 8 = L$
$L-\varepsilon = L-2 = 6$
$\Bigg\}$ Epsilon tube
ε-tube

The idea is to keep $|x-3| < \delta = \frac{2}{3}$

That means

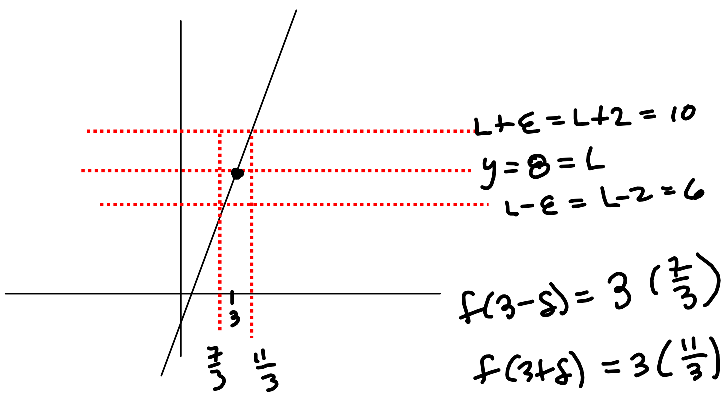
$-\delta < x-3 < \delta$

$\Rightarrow 3-\delta < x < 3+\delta$

since $\delta = \frac{2}{3}$, this means

$3-\frac{2}{3} < x < 3+\frac{2}{3}$

$\frac{7}{3} < x < \frac{11}{3}$



$f(3-\delta) = 3\left(\frac{7}{3}\right) - 1 = 7-1 = 6 = f\left(\frac{7}{3}\right)$

$f(3+\delta) = 3\left(\frac{11}{3}\right) - 1 = 11-1 = 10 = f\left(\frac{11}{3}\right)$

Summarize:

$|f(x)-L| = |f(x)-8| = |3x-1-8| = |3x-9| = 3|x-3| < 3\left(\frac{2}{3}\right) = 2$

whenever $|x-3| < \frac{2}{3}$

It's a limit so we never let $x=3$:

$0 < |x-3| < \frac{2}{3}$

Claim

To keep $3x-1$ less than $\varepsilon = 2$ units from $L=8$, keep x less than $\frac{2}{3}$ units from $x=3$.

Proof

If $0 < |x-3| < \frac{2}{3}$, then $|f(x)-8| = |3x-1-8| = |3x-9| = 3|x-3|$ $< 3\left(\frac{2}{3}\right) = 2 = \varepsilon$ & we are done (since this guarantees $|f(x)-L| = |3x-1-8| < 2$.)

$\rightarrow$ $x=3$ not allowed!

What about $\varepsilon = 0.1$?

Scratch: want $|3x-1-8| < 0.1 = \frac{1}{10}$

$\Longrightarrow 3|x-3| < \frac{1}{10}$

$\Longrightarrow |x-3| < \frac{1}{3} \cdot \frac{1}{10} = \frac{1}{30} \equiv \delta$

Claim:

$0 < |x-3| < \frac{1}{30} \Longrightarrow |f(x)-L| < 0.1$

Proof

$0 < |x-3| < \frac{1}{30} \Longrightarrow |f(x)-L| = |3x-1-8| = |3x-9| = 3|x-3|$ $< 3\delta = 3\left(\frac{1}{30}\right) = \frac{1}{10}$, so we're done. ∎

$\lim_{x \to c} f(x) = L$ means

Given any $\varepsilon > 0$, there is a $\delta > 0$ such that
$0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$

Prove $\lim_{x \to 3} (3x-1) = 9$.

Scratch: want $|f(x) - L| = |3x-1-9| = 3|x-3| < \varepsilon \Rightarrow$
$|x-3| < \frac{\varepsilon}{3}$. Define $\boxed{\delta = \frac{\varepsilon}{3}.}$

Write proof!

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{3}$. Then
$0 < |x-3| < \delta \Rightarrow |f(x) - L| = |3x-1-8| = |3x-9|$
$= 3|x-3| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$ ■

Math shorthand
- such that or so that $\ni$
- Implies $\Rightarrow$
- $\exists$ - There is or There exists
- $\forall$ - For each or for all or for every

Claim

$$\lim_{x \to 5} (7x-5) = 30 \quad . \quad f(x) = 7x-5$$

Proof

Let $\epsilon > 0$ be given. Define $\delta = \frac{\epsilon}{7}$. Then

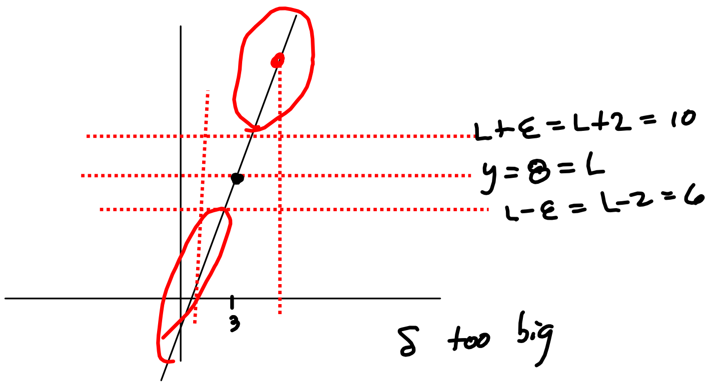
$0 < |x-5| < \delta \implies$

$|f(x) - L| = |7x-5-30| = |7x-35| = 7|x-5| < 7\delta = 7\left(\frac{\epsilon}{7}\right) = \epsilon \ \blacksquare$

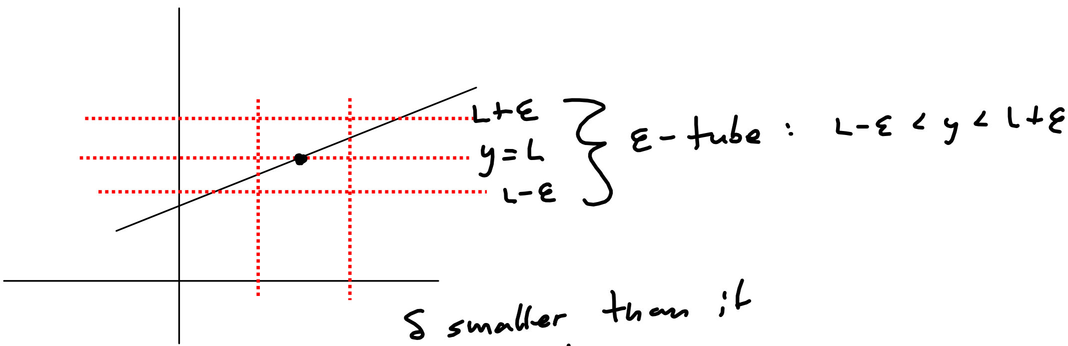
If m is slope/growth rate giving change in y with respect to unit change in x, then

$\delta = \frac{\epsilon}{m}$, every time.

We want to keep $f(x)$ from leaking out the bottom or top of the "Epsilon tube" (ϵ tube)



δ too big



δ smaller than it needs to be

Tomorrow:

The next level is handling a quadratic.

Let $f(x) = x^2 - 3x - 5$. Then

$\lim_{x \to 2} f(x) = -7$.

The idea:

Scratch want $|f(x) - L| = |x^2 - 3x - 5 - (-7)|$

$= |x^2 - 3x + 2| = |x-1| \underbrace{|x-2|}_{< \delta}$

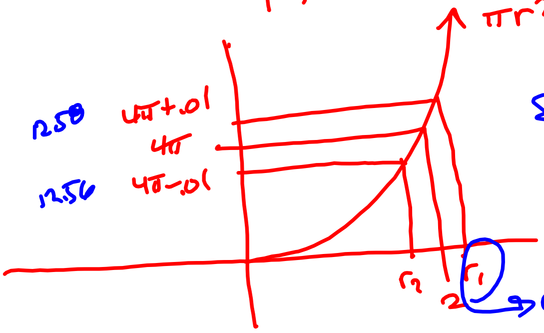
We'll need a bound on this factor.

What's the tolerance on the radius of a disc, to keep its area within .01 of 4π? $(2^2\pi)$

Area $= \pi r^2$

Want $|\text{Area} - 4\pi| < \varepsilon = .01$

$= |\pi r^2 - 4\pi| < .01$



closer than r_2 because $f(r)$ is steeper to the right

δ = smaller of $|r_1 - r|$ & $|r_2 - r|$

12.5763706144 $- 4\pi$

Solve $\pi r^2 = 4\pi + .01$

$r^2 = \frac{1}{\pi}[4\pi + .01]$

$r_1 = \pm\sqrt{\frac{1}{\pi}[4\pi + .01]}$ Take $r > 0$ ("+")

plug it in

$\pi r^2 = 4\pi - .01$

$r^2 = \frac{1}{\pi}[4\pi - .01]$

$r_2 = \pm\sqrt{\frac{1}{\pi}[4\pi - .1}$

plug it in

Claim $f(x) = x^2 - 5x + 3 \Rightarrow$ $\begin{array}{r} 5^2 \\ -35 \end{array}$

$\lim_{x \to 7} f(x) = 49 - 35 + 3 = 17$

Scratch want $|f(x) - L| = |x^2 - 5x + 3 - 17|$

$= |x^2 - 5x - 14| = |x-7||x+2| < \delta|x+2|$

Need a ceiling/bound on $|x+2|$

Start by assuming $\delta \leq 1$.

Since $x \to 7$, this means

This is bad.

$6 < x < 1 \to 8$ idiot $\Rightarrow$

This should be $|x+2|$ from here

$6+1 < x+1 < 1+1 \to 8 \Rightarrow$

$7 < x+1 < 2 \Rightarrow$

$|x+1| < 7$

The heart of it
Build $m < x+1 < m$
from $\_ < x < \_$

Then $|x+1|\delta < 7\delta$.

Define $\delta =$ minimum of 1 & $\frac{\varepsilon}{7}$

$\delta = \min\{1, \frac{\varepsilon}{7}\}$

Nexup.

~~**Proof**~~

~~Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{7}\}$.~~

~~Then $0 < |x-7| < \delta \Rightarrow |f(x) - L| = |x^2 - 5x + 3 - 17|$~~

~~$= |x^2 - 5x - 14| = |x+1||x-7|$~~

~~$< |x+1|\delta < 7\delta \leq 7\left(\frac{\varepsilon}{7}\right) = \varepsilon$~~

~~DONE!~~

$\delta \le 1 \longrightarrow$

$6 < x < 8 \longrightarrow$

This is good.

$8 < x+2 < 10 \longrightarrow$

Let $\delta = \cancel{\frac{\varepsilon}{10}} \min\{1, \frac{\varepsilon}{10}\}$.

Then $0 < |x-7| < \delta \longrightarrow$

$|f(x) - L| = |x^2 - 5x + 3 - 17|$

$= |x^2 - 5x - 14| = |x-7||x+2|$

$< \delta |x+2| < \delta \cdot 10 = 10\delta \le 10\left(\frac{\varepsilon}{10}\right) = \varepsilon$ ∎

That's better.

This is <u>Bonus</u>

Should've waited until tomorrow. Hulk exhausted. Been up grading 1340 all night.

Needs a nap. Sun is getting low...