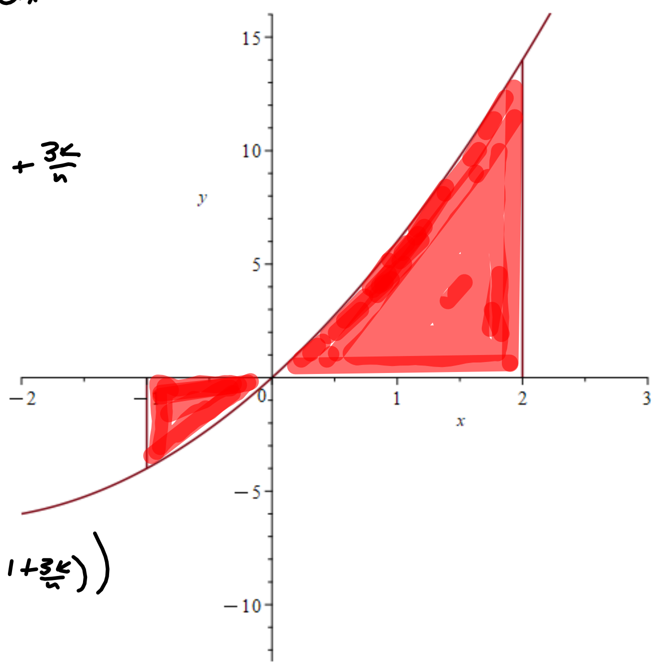
1. (10 pts) Use the limit definition of the definite integral to evaluate $\int_{-1}^{2}\left(x^2+5x\right)dx$. Use a right-endpoint Riemann sum. I don't want you to take it all the way, but I do expect to see the $\Delta x, x_k, f\left(x_k\right)$ written explicitly. Stop just short of actually passing to the limit.

1 $\int_{-1}^{2}\left(x^2+5x\right)dx = \int_a^b f(x)dx$



$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_k = a + k\Delta x = -1 + k\cdot\frac{3}{n} = -1 + \frac{3k}{n}$

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$:

$\sum_{k=1}^{n} f(x_k)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k) =$

$= \frac{3}{n}\sum_{k=1}^{n} f\left(-1+\frac{3k}{n}\right) = \frac{3}{n}\sum_{k=1}^{n}\left(\left(-1+\frac{3k}{n}\right)^2 + 5\left(-1+\frac{3k}{n}\right)\right)$

$= \frac{3}{n}\sum_{k=1}^{n}\left(1 - \frac{6k}{n} + \frac{9k^2}{n^2} - 5 + \frac{15k}{n}\right)$

$= \frac{3}{n}\sum_{k=1}^{n}\left(-4 + \frac{9k^2}{n^2} + \frac{9k}{n}\right)$

$= \frac{3}{n}\left[-\sum_{k=1}^{n} 4 + \frac{9}{n^2}\sum_{k=1}^{n} k^2 + \frac{9}{n}\sum k\right]$

$= \frac{3}{n}\left[-4n + \frac{9}{n^2}\left(\frac{2n^3+\cdots}{6}\right) + \frac{9}{n}\left(\frac{n^2+\cdots}{2}\right)\right] = -12 + \frac{27}{n^3}\left(\frac{2n^3+\cdots}{6}\right) + \frac{27}{n^2}\left(\frac{n^2+\cdots}{2}\right)$

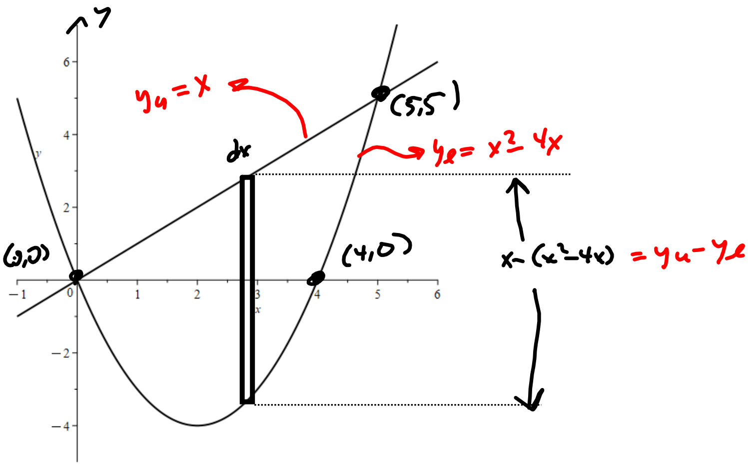
Bonus (5 pts bonus) Pass to the limit in your answer to #1.

$\xrightarrow{n\to\infty} -12 + \frac{54}{6} + \frac{27}{2} = -12+9+\frac{27}{2} = -3+\frac{27}{2} = \frac{-6+27}{2} = \boxed{\frac{21}{2}}$

2. Find the area of the region bounded by $y = x^2 - 4x$ and $y = x$. in two ways.

   a. (5 pts) Sketch the region.

(a) 5pts



b. (5 pts) Write the area as an integral with respect to x. Draw a representative rectangle on the sketch from part a.

$\int_a^b (y_u - y_l)\,dx$

(b) 5pts $\boxed{\text{Area} = \int_0^5 \left(x - (x^2 - 4x)\right)dx}$

c. (5 pts) Evaluate the integral from part b.

c 5pts Area $= \int_0^5 (5x - x^2)\,dx$

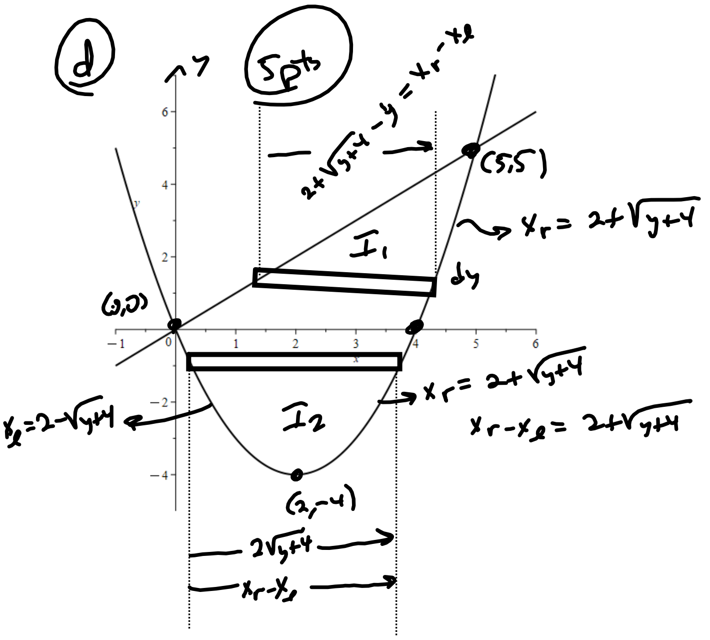
$$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = \frac{5(5)^2}{2} - \frac{5^3}{3}$$

$$= \frac{5^3(3) - 5^3(2)}{6} = \frac{5^3(3-2)}{6}$$

$$= \frac{5^3}{6} = \boxed{\frac{125}{6}}$$

d. (5 pts) Sketch the region again.

d



$x_r - x_l = 2+\sqrt{y+4} - 2 - \sqrt{y+4} = 2\sqrt{y+4}$!

Inverting should be here, but the way I worked it, a lot of that work is done in part d. so not the best wording by me.

e. (5 pts) Write the area as the sum of two integrals with respect to y. Draw representative rectangles. There will be two different regions, so you will need a rectangle for each region.

e 5pts Invert $f(x)$ & $g(x)$

$y = x \longrightarrow x = y$

$y = x^2 - 4x = x^2 - 4x + 2^2 - 4$

$= (x-2)^2 - 4 = y \longrightarrow$

$(x-2)^2 = y + 4 \longrightarrow$

$x - 2 = \pm\sqrt{y+4} \longrightarrow$

$x = 2 \pm \sqrt{y+4}$

I_1: $x_r = 2 + \sqrt{y+4}$

$x_\ell = y$

$$\boxed{I_1 = \int_0^5 \left(2 + \sqrt{y+4} - y\right) dy}$$

I_2: $x_r = 2 + \sqrt{y+4}$

$-\ (x_\ell = 2 - \sqrt{y+4})$

$x_r - x_\ell = 2\sqrt{y+4}$

$$\boxed{I_2 = \int_{-4}^{0} 2\sqrt{y+4}\, dy}$$

$$\boxed{\text{Area} = I_1 + I_2}$$

I went back to 2d to supply more details

I should have been more specific in the wording of part d.

To invert a function, simply solve y = f(x) for x:

Method 1: Complete the Square (on the left).

Method 2: Use the Quadratic Formula below:

$y = x^2 - 4x \longrightarrow$

$x^2 - 4x - y = 0 \longrightarrow$

$a = 1, b = -4, c = -y$

$b^2 - 4ac = 16 - 4(1)(-y) = 16 + 4y = 4y + 16$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{4y+16}}{2} = \frac{4 \pm \sqrt{4(y+4)}}{2}$

$= \frac{4 \pm 2\sqrt{y+4}}{2} = \frac{2\left(2 \pm \sqrt{y+4}\right)}{2}$

$= 2 \pm \sqrt{y+4}$

$\cdot 2 - \sqrt{y+4}$, $x_r = 2 + \sqrt{y+4}$

SEE SKETCH, PREVIOUS PAGE.

f. (5 pts) Evaluate the sum of integrals from part e.

$$I_1 = \int_0^5 \left(2 + \sqrt{y+4} - y\right) dy = \left[2y + \frac{2}{3}(y+4)^{\frac{3}{2}} - \frac{y^2}{2}\right]_0^5$$

$$= 2(5) + \frac{2}{3}(9)^{\frac{3}{2}} - \frac{25}{2} - \left(\frac{2}{3}(4)^{\frac{3}{2}}\right)$$

$$= 10 + \frac{2}{3}(3)^3 - \frac{25}{2} - \left(\frac{2}{3}\right)(2)^3 = 10 + 18 - \frac{25}{2} - \frac{16}{3}$$

$$= 28 - \frac{75}{6} - \frac{32}{6} = 28 - \frac{107}{6} = \frac{168 - 107}{6} = \boxed{\frac{61}{6} = I_1}$$

$$I_2 = \int_{-4}^{0} 2\sqrt{y+4}\, dy = 2\left(\frac{2}{3}\right)(y+4)^{\frac{3}{2}}\Big]_{-4}^{0}$$

$$= \frac{4}{3}\cdot(y+4)^{\frac{3}{2}}\Big]_{-4}^{0} = \frac{4}{3}(4)^{\frac{3}{2}} = \frac{4}{3}(2)^3 = \boxed{\frac{32}{3} = I_2}$$

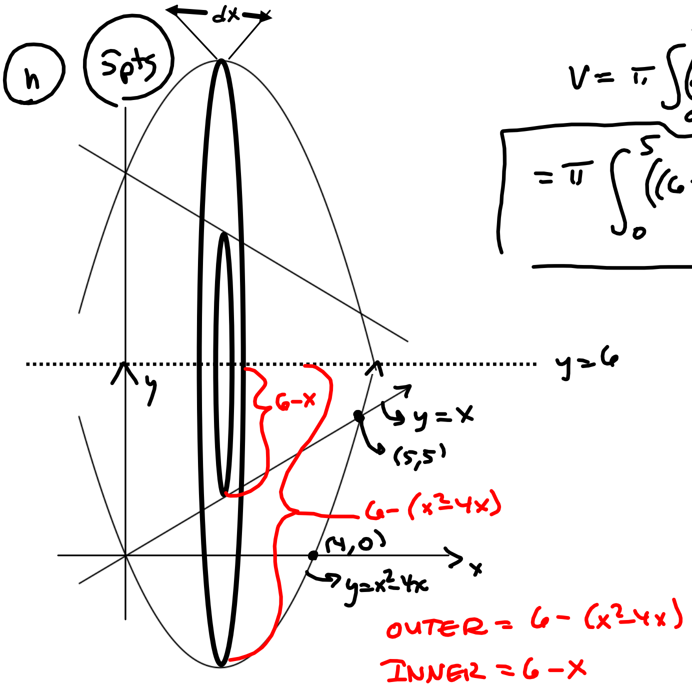
$$\text{Area} = I_1 + I_2 = \frac{61}{6} + \frac{32}{3} = \frac{61+64}{6} = \boxed{\frac{125}{6} = \text{AREA}}$$ 5pts f

g. (5 pts bonus) Compare your results from parts c and f.

3g 5pts The areas were the same (good sign).
Integrating wrt 'y' was more difficult, by far.

h. (5 pts) Suppose we rotated the region about the line $y = 6$. Sketch the graph, and write the integral representing the volume of the solid of revolution obtained. Show a representative disc or washer.

h 5pts



$$V = \pi \int_0^5 (\text{OUTER}^2 - \text{INNER}^2)\,dx$$

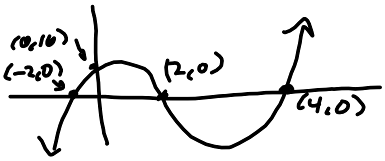
$$\boxed{= \pi \int_0^5 \left((6-(x^2-4x))^2 - (6-x)^2 \right) dx}$$

3. We explore absolute value. Let $f(x) = x^3 - 4x^2 - 4x + 16$

a. (5 pts) Provide a rough sketch of $f(x)$.

$f' = 3x^2 - 8x - 4$
$= 3(x^2 - \frac{8}{3}x + (\frac{4}{3})^2) - 3(\frac{16}{9}) - 4$
$= 3(x - \frac{4}{3})^2 - \frac{20}{3} = 0$
$x = \frac{4}{3} \pm \frac{2\sqrt{5}}{3}$

$f(x) = x^3 - 4x^2 - 4x + 16$
$= x^2(x-4) - 4(x-4)$
$= (x-4)(x^2-4)$
$= (x-4)(x-2)(x+2)$



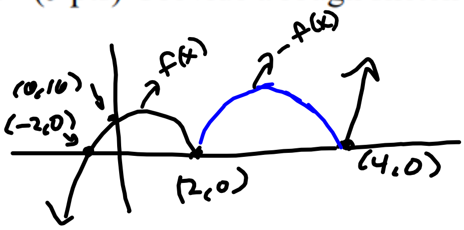
b. (5 pts) Evaluate $\int_0^4 f(x)\,dx$. $\int_0^4 (x^3 - 4x^2 - 4x + 16)\,dx = F(4) - F(0) = F(4)$

$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - 2x^2 + 16x\right]_0^4 = 64 - \frac{256}{3} - 32 + 64 = \frac{96(3)}{3} - \frac{256}{3}$

$\frac{288 - 256}{3} = \boxed{\frac{32}{3}} = \int f$

288

c. (5 pts) Provide a rough sketch of $y = |f(x)|$.



$$|f(x)| = \begin{cases} f(x) & \text{if } 0 \le x \le 2 \\ -f(x) & \text{"} \quad 2 < x \le 4 \end{cases}$$

d. (5 pts) Evaluate $\int_0^4 |f(x)|\,dx$.

$= \int_0^2 f(x)\,dx - \int_2^4 f(x)\,dx = F(2) - F(0) - (F(4) \quad F(2))$
$= 2F(2) - F(4) = 2F(2) - \frac{32}{3}$

From b.

$= 2\left[\frac{2^4}{4} - \frac{4(2)^3}{3} - 2(2)^2 + 16(2)\right] - \frac{32}{3}$

$= 2\left[4 - \frac{32}{3} - 8 + 32\right] - \frac{32}{3} = 2\left[28 - \frac{32}{3}\right] - \frac{32}{3}$

$= 56 - \frac{96}{3} = \frac{168 - 96}{3} = \frac{72}{3} = \boxed{24} = \int |f|$

I calculated it in a strange way that was a little easier for me. The *F*(*x*) in the above is the antiderivative of *f*(*x*) (the one with the constant of integration = 0).

4. Evaluate the indefinite integrals:

a. (5 pts) $\int (3x+2)^3\,dx = \int u^3 \frac{du}{3}$, where $u = 3x+2$ &

$du = 3dx$

$= \frac{1}{3}\cdot\frac{1}{4}u^4 + C = \boxed{\frac{(3x+2)^4}{12} + C}$

b. (5 pts) $\int x^2(3x+2)^4\,dx$

Let $u = 3x+2$. Then $du = 3dx \Rightarrow dx = \frac{du}{3}$

& $u-2 = 3x \Rightarrow$

$x = \frac{u-2}{3} \Rightarrow$

$\int \left(\frac{u-2}{3}\right)^2 (u^4)\,\frac{du}{3}$

$= \frac{1}{3^3}\int (u^2 - 4u + 4)u^4\,du = \frac{1}{27}\int (u^6 - 4u^5 + 4u^4)\,du$

$= \frac{1}{27}\left[\frac{u^7}{7} - \frac{4u^6}{6} + \frac{4u^5}{5}\right] + C =$

$= \boxed{\frac{1}{27}\left[\frac{(3x+2)^7}{7} - \frac{2(3x+2)^6}{3} + \frac{4(3x+2)^5}{5}\right] + C}$

c. (5 pts) $\int \sin^4(x)\cos(x)\,dx = \int u^4\,du$, where $u = \sin(x)$ &

$du = \cos(x)\,dx$

$= \boxed{\frac{\sin^5(x)}{5} + C}$

d. (5 pts) $\int \sin(x)\cdot 2^{\cos(x)}\,dx = -\int \left(2^{\cos(x)}\right)(-\sin(x)dx)$

$= -\int 2^u\,du = -\frac{1}{\ln(2)}\cdot 2^u + C = \boxed{\frac{-1}{\ln(2)}\left(2^{\cos(x)}\right) + C}$

5. Perform the indicated differentiation:

a. (5 pts) $\dfrac{d}{dx}\displaystyle\int_0^x \dfrac{\cos(2t+1)}{t^2-7}dt = \boxed{\dfrac{\cos(2x+1)}{x^2-7}}$

b. (5 pts) $\dfrac{d}{dx}\displaystyle\int_{\sin(x)}^x \dfrac{\sin(3t)}{t^2+4}dt = \dfrac{d}{dx}\left[\int_{\sin(x)}^0 \sim + \int_0^x \sim\right]$

$$= \frac{d}{dx}\left[-\int_0^{\sin(x)} \sim + \int_0^x \sim\right]$$

$$= \boxed{-\left(\frac{\sin(3\sin(x))}{\sin^2(x)-7}\right)(\cos(x)) + \frac{\sin(3x)}{x^2-7}}$$

6. The function $f(x) = x^2 - 4x$ is 1-to-1 on the restricted domain $D = [2, \infty)$.

a. (10 pts) Find the inverse function $f^{-1}(x)$. State its domain and range.

$y = x^2 - 4x \rightsquigarrow y^2 - 4y = x \longrightarrow$

$y^2 - 4y + 2^2 = x + 4 \longrightarrow$

$(y-2)^2 = x + 4 \longrightarrow$

$y - 2 = \pm\sqrt{x+4}$

$y = 2 \pm \sqrt{x+4}$ want $R(f^{-1}) = D(f) = [2, \infty)$

$\longrightarrow \boxed{f^{-1}(x) = 2 + \sqrt{x+4}} = 2 + (x+4)^{\frac{1}{2}}$

b. (5 pts) Find $(f^{-1})'(5)$, directly, by differentiating your answer for part a.

$(f^{-1})'(x) = \frac{1}{2}(x+4)^{-\frac{1}{2}} \longrightarrow (f^{-1})'(5) = \frac{1}{2}(5+4)^{-\frac{1}{2}} = \frac{1}{2}(9)^{-\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{3}\right) =$

$= \boxed{\frac{1}{6} = (f^{-1})'(5)}$

c. (5 pts) Find $(f^{-1})'(5)$ by applying a theorem regarding derivatives of inverse functions.

$f(x) = x^2 - 4x \overset{\text{SET}}{=} 5 \longrightarrow$

$x^2 - 4x - 5 = (x-5)(x+1) = 0 \longrightarrow x = -1, 5$

Restricted domain $= [2, \infty) \longrightarrow f^{-1}(5) = 5$

$f'(x) = 2x - 4 \longrightarrow$

$(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{2(5) - 4} = \dfrac{1}{10-4} = \boxed{\frac{1}{6} = (f^{-1})'(5)}$ ✓

7. (5 pts each) Find the derivative with respect to x.

a. $y = 5 \cdot 7^{x^2+5x}$ $\rightarrow$ $\boxed{y' = 5\ln(7)\left(7^{x^2+5x}\right)(2x+5)}$

b. $y = \ln\left(\frac{(7x^3-8)^5}{\sqrt{2x\sin(x)}}\right) = 5\ln(7x^3-8) - \frac{1}{2}\left[\ln x + \ln(\sin(x))\right]$

$\rightarrow$ $\boxed{y' = \frac{5(21x^2)}{7x^3-8} - \frac{1}{2}\left(\frac{1}{x}\right) - \frac{1}{2}\frac{\cos(x)}{\sin(x)}}$

c. $y = \log_7(x^2-3x)$ $\rightarrow$ $\boxed{y' = \frac{1}{\ln(7)}\left(\frac{2x-3}{x^2-3x}\right)}$

d. $y = [\tan(x)]^{x^2+4x}$ $\rightarrow$ $\ln(y) = (x^2+4x)\ln(\tan(x))$ $\rightarrow$

$\boxed{y' = \left((2x+4)(\ln(\tan(x)) + (x^2+4x)\left(\frac{\sec^2(x)}{\tan(x)}\right)\right)\left(\tan(x)^{x^2+4x}\right)}$

Bonus 1 (5 pts) Confirm that the hypotheses of the Mean Value Theorem hold for $f(x) = x^3 - 2x^2 + 5x - 1$ on $[0,3]$, and find the c that is promised in the conclusion of the theorem.

f is a polynomial; $\therefore$ f is cnt^s & difb^l $\forall x \in \mathbb{R}$, hence cnt^s on $[0,3]$ & difb^l on $(0,3)$.

$\therefore \exists c \in (0,3) \ni f'(c) = 3c^2 - 4c + 5 = \frac{f(3)-f(0)}{3-0} = \frac{23-(-1)}{3} = 8$

$\Rightarrow 3c^2 - 4c - 3 = 0 \longrightarrow$

$3\left(c^2 - \frac{4}{3}c + \left(\frac{2}{3}\right)^2\right) - 3\left(\frac{4}{9}\right) - 3$

$= 3\left(c - \frac{2}{3}\right)^2 - \frac{4}{3} - \frac{9}{3} = 3\left(c - \frac{2}{3}\right)^2 - \frac{13}{3} = 0$

$\Rightarrow \left(c - \frac{2}{3}\right)^2 = \frac{13}{9}$

$\Rightarrow c = \frac{2}{3} \pm \frac{\sqrt{13}}{3} \Rightarrow \boxed{c = \frac{2+\sqrt{13}}{3}} \approx 1.868517092$

Scratch:

3	1	-2	5	-1
		3	3	24
	1	1	8	23

Bonus 2 (5 pts) Use the tangent line to approximate $\cos(33^0)$.

$f(x) = \cos(x)$

$x_1 = 30^0 = \frac{\pi}{6}$ *

$f(x_1) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\longrightarrow f'(x) = -\sin(x)$

$\longrightarrow f'(x_1) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$\Rightarrow L(x) = f'(x_1)(x - x_1) + f(x_1)$

$= -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$

$\Rightarrow L(33^0) = L\left(33^0\left(\frac{\pi}{180^0}\right)\right) = L\left(\frac{33\pi}{180^0}\right)$

$= -\frac{1}{2}\left(\frac{33\pi}{180^0} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(\frac{3\pi}{180}\right) + \frac{\sqrt{3}}{2} = \boxed{\frac{-\pi}{120} + \frac{\sqrt{3}}{2}}$

$\approx$ $\approx \cos(33^0)$

*Why convert to radians?

Because $\frac{d}{dx}[\cos(x)] = -\sin(x)$ only if x is measured in radians. This is needed in the proof of $\frac{d}{dx}[\cos(x)] = -\sin(x)$, when we equate the angle h and the arc length of the arc corresponding to h.

This arc length is also h, on the unit circle

Angle h = arc length

In degrees, angle h corresponds to arc length $s = \frac{h \cdot \pi}{180^\circ}$!

So there's a nasty $\frac{\pi}{180^\circ}$ factor in the proof, and we're not going to fool with that.

Working this in degrees (incorrectly):

$\Rightarrow f(x_1) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$ $\qquad -\sin(30^\circ) = -\frac{1}{2}$

$L_{x_1}(x) = f'(x_1)(x - x_1) + f(x_1)$

$= -\frac{1}{2}(33^\circ - 30^\circ) + \frac{\sqrt{3}}{2} = -\frac{3}{2} + \frac{\sqrt{3}}{2} \approx -.6339745960$

NOT close to $\cos(30^\circ)$, which it should be.

Here's the derivative of cosine, using degrees:

$$\frac{d}{dx}[\cos(x^\circ)] = \frac{d}{dx}\left[\cos\left((x^\circ)\left(\frac{\pi}{180^\circ}\right)\right)\right] =$$

$$= -\frac{\pi}{180^\circ}\sin(x^\circ) \ !$$

If you use this for the derivative of cosine, then it all works out just fine.

Bonus 3 (5 pts) Find $\frac{dy}{dx}$ if $x^2 - 3xy + y^2 = 1$. Then find an equation of the tangent line to the curve at $(1,3)$.

$\Rightarrow 2x - 3y - 3xy' + 2yy' = 0$

$\Rightarrow (-3x+2y)y' = -2x+3y$

$\Rightarrow y' = \frac{-2x+3y}{-3x+2y} \Rightarrow$

$$y'\Big|_{(x,y)=(1,3)} = \frac{-2(1)+3(3)}{-3(1)+2(3)} = \frac{9-2}{-3+6} = \boxed{\frac{7}{3} = y'\Big|_{(x,y)=(1,3)}}$$

Bonus 4 (5 pts) Evaluate the integral for #2h. You only get credit if your #2h is correct.

$$=\pi\int_0^5 \left((6-(x^2-4x))^2-(6-x)^2\right)dx$$

$$=\pi\int_0^5 (x^2-4x)^2-12(x^2-4x)+36-(x^2-12x+36))\,dx$$

$$=\pi\int_0^5 x^4-8x^3+16x^2-12x^2+48x+36-x^2+12x-36)\,dx$$

$$=\pi\int_0^5 (x^4-8x^3+3x^2+60x)\,dx=\pi\left[\frac{x^5}{5}-\frac{8x^4}{4}+\frac{3x^3}{3}+\frac{60x^2}{2}\right]_0^5$$

$$=\pi\left[\frac{x^5}{5}-2x^4+x^3+30x^2\right]_0^5=\pi\left[5^4-2(5^4)+5^3+30(5^2)\right]$$

$$=\pi\left[5^2(5^2-2(5^2)+5+30)\right]=\pi(25(-25+35))=\pi(25(10))=\boxed{250\pi}$$

Bonus 5 Prove that $\lim_{x\to -3}\left(x^2-2x+5\right)=20 = L$

Let $f(x) = x^2-2x+5$.

WTS $\forall \varepsilon>0, \exists \delta>0 \ni |x-(-3)|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

Scratch: $|x-(-3)| = |x+3| < \delta$

$|f(x)-L| = |x^2-2x+5-20| = |x^2-2x-15|$

$= |x-5||x+3| = |x-5||x-(-3)| < |x-5|\delta$.

Assume $\delta \le 1$. Then

$0 < |x+3| < \delta \le 1 \Rightarrow$

$|x+3| < 1 \Rightarrow$

$-1 < x+3 < 1$ want $x-5$:

$-8 = -8 = -8$

$-9 < x-5 < -7$

$\Rightarrow |x-5| < 9 \Rightarrow$ Need $\delta = \min\{1, \frac{\varepsilon}{9}\}$

$\boxed{\text{Proof of original Claim:}}$

Let $\varepsilon>0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{9}\}$

Then $0<|x-(-3)|<\delta \Rightarrow |f(x)-L| = |x^2-2x+5-20|$

$= |x^2-2x-15| = |x-5||x+3| < 9\delta \le 9\left(\frac{\varepsilon}{9}\right) = \varepsilon$ ■