1. Ch 4 (10 pts) Use the limit definition of the definite integral to evaluate $\int_{-1}^{2}\left(2x^2 - x\right)dx$. For simplicity, use the limit of the right-endpoint Riemann sum. On the final, I'll be looking for the correct Riemann Sum. Evaluating it will be bonus.

$a=-1,\ b=2$

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_k = a + k\Delta x = -1 + \frac{3k}{n}$

$f(x_k) = 2x_k^2 - x_k = 2\left(-1+\frac{3k}{n}\right)^2 - \left(-1+\frac{3k}{n}\right) -$

$= 2\left(1 - \frac{6k}{n} + \frac{9k^2}{n^2}\right) + 1 - \frac{3k}{n} = 2 - \frac{12k}{n} + \frac{18k^2}{n^2} + 1 - \frac{3k}{n}$

$= 3 - \frac{15k}{n} + \frac{18k^2}{n}$

$\Delta x \sum f(x_k) = \frac{3}{n}\sum_{k=1}^{n}\left(3 - \frac{15k}{n} + \frac{18k^2}{n^2}\right)$

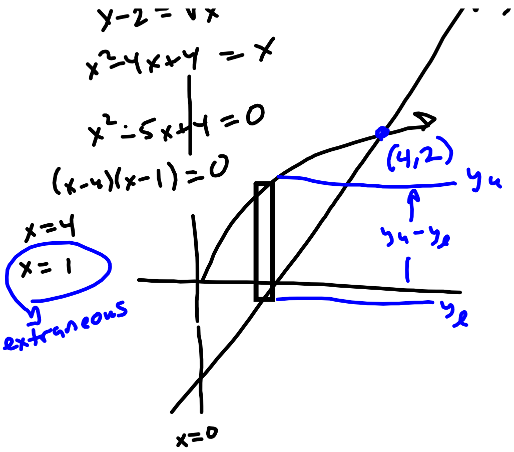
$= \frac{3}{n}\left[\sum_{k=1}^{n} 3 - \frac{15}{n}\sum_{k=1}^{n} k + \frac{18}{n^2}\sum_{k=1}^{n} k^2\right]$

$= \frac{9}{n}\cdot n - \frac{3}{n}\cdot\frac{15}{n}\left(\frac{n^2+\cdots}{2}\right) + \frac{3}{n}\cdot\frac{18}{n^2}\left(\frac{n^3+\cdots}{3}\right)$

$\xrightarrow{n\to\infty} 9 - \frac{45}{2} + 18 = \frac{18-45+36}{2} = \boxed{\frac{9}{2}}$

2. We find the area of the region bounded by $y=\sqrt{x}, y=x-2,$ and $x=0$ in two ways.

a. Ch 3 (5 pts) Sketch the region.



Area bdd by $y=\sqrt{x}$, $y=x-2$, $x=0$

b. Ch 4 (5 pts) Write the area as an integral with respect to x. Draw a representative rectangle on the sketch from part a.

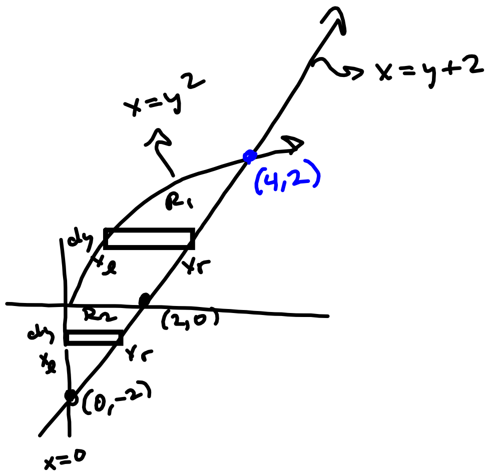
$$\text{AREA} = \int_a^b (y_u - y_\ell)\,dx = \int_0^4 (\sqrt{x} - (x-2))\,dx$$

c. Ch 4 (5 pts) Evaluate the integral from part b.

$$= \int_0^4 (x^{\frac{1}{2}} - x + 2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_0^4 = \frac{2}{3}(4)^{\frac{3}{2}} - \frac{4^2}{2} + 2(4) - (0)$$

$$= \frac{2}{3}(8) - 8 + 8 = \boxed{\frac{16}{3} = \text{Area}}$$

d. Ch 3 (5 pts) Sketch the region again.



e. Ch 4 (5 pts) Write the area as an integral with respect to y. Draw a representative rectangle on the sketch from part d.

$y=\sqrt{x}$, $x=y^2$ $\qquad$ $y=x-2$, $x=y+2$

$R_1: \int_a^b (x_r - x_l)\,dy = \int_0^2 (y+2-y^2)\,dy$

$R_2: \int_a^b (x_r - x_l) = \int_{-2}^0 (y+2-0)\,dy$

$$A = R_1 + R_2 = \int_0^2 (y+2-y^2)\,dy + \int_{-2}^0 (y+2)\,dy$$

f. Ch 4 (5 pts) Evaluate the integral from part e.

$$\text{Area} = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 + \left[\frac{y^2}{2} + 2y\right]_{-2}^0$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - (0) + (0) - \left(\frac{(-2)^2}{2} + 2(-2)\right)$$

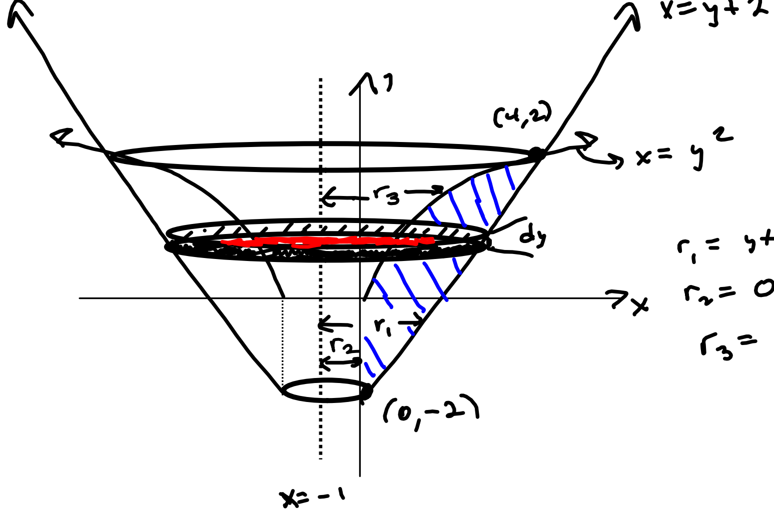
$$= 2 + 4 - \frac{8}{3} - 2 + 4 = 8 - \frac{8}{3} = \frac{24-8}{3} = \boxed{\frac{16}{3} = \text{Area}}$$

g. Ch 4 (5 pts bonus) Compare your results from parts c and f.

Part c was a lot easier. I got the same result, though, so I'm *very* confident of my result.

h. Ch 5 (5 pts) Suppose we rotated the region about the line $x = -1$. Sketch the graph, and write the integral representing the volume of the solid of revolution obtained. Show a representative disc or washer.

Volume : $x = -1$



Volume =

$$\int_{-2}^{0} (r_1^2 - r_2^2)\,dy + \int_0^2 (r_1^2 - r_3^2)\,dy$$

$r_1 = y+2-(-1) = y+3$

$r_2 = 0-(-1) = 1$

$r_3 = y^2-(-1) = y^2+1$

h. Ch 5 (5 pts) Suppose we rotated the region about the line $x = -1$. Sketch the graph, and write the integral representing the volume of the solid of revolution obtained. Show a representative disc or washer.

$$V = \pi\int_{-2}^{0}\left((y+3)^2 - (1)^2\right)dy + \pi\int_{0}^{2}\left((y+3)^2 - (y^2+1)^2\right)dy$$

$$\pi\int_{-2}^{0}(y^2+6y+9-1)\,dy + \pi\int_{0}^{2}\left(y^2+6y+9-(y^4+2y^2+1)\right)dy$$

$$= \pi\left[\frac{y^3}{3}+\frac{6y^2}{2}+8x\right]_{-2}^{0} + \pi\int_{0}^{2}(y^2+6y+9-y^4-2y^2-1)\,dy$$

$$= \pi\left[0-\left(\frac{(-2)^3}{3}+3(-2)^2+8(-2)\right)\right] + \pi\int_{0}^{2}(-y^4-y^2+6y+8)\,dy$$

$$= \pi\left[-\left(-\frac{8}{3}+12-16\right)\right] + \pi\left[-\frac{y^5}{5}-\frac{y^3}{3}+\frac{6y^2}{2}+8y\right]_{0}^{2}$$

$$\begin{array}{r} 280 \\ 140 \\ \hline 420 \end{array}$$

$$= \pi\left[-\left(-\frac{8}{3}-4\right)\right] + \pi\left[-\frac{2^5}{5}-\frac{2^3}{3}+3(2^2)+8(2)-0\right]$$

$$= \pi\left[\frac{8+12}{3}\right] + \pi\left[-\frac{32}{5}-\frac{8}{3}+12+16\right] = \pi\left[\frac{20}{3}\right] + \pi\left[\frac{-96-40+28(15)}{15}\right]$$

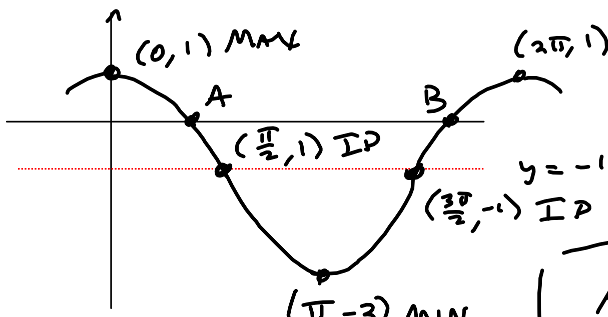
$$= \frac{20\pi}{3} + \pi\left[\frac{-136+420}{15}\right] = \frac{20\pi}{3} + \pi\left[\frac{284}{15}\right]$$

$$= \frac{100\pi+284\pi}{15} = \frac{384\pi}{15} = \boxed{\frac{128\pi}{5}}$$

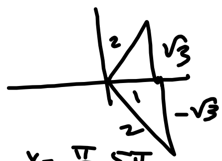
3. We explore absolute value. Let $f(x) = 2\cos(x) - 1$

a. Ch 3 (5 pts) Sketch a complete graph of $f(x)$ on the interval $[0, 2\pi]$.

Don't even need calc! Inflection pts on the midline, $y = -1$
max's & mins are $y = -1 \pm 2$



$2\cos(x) - 1 = 0$
$\cos(x) = \frac{1}{2}$



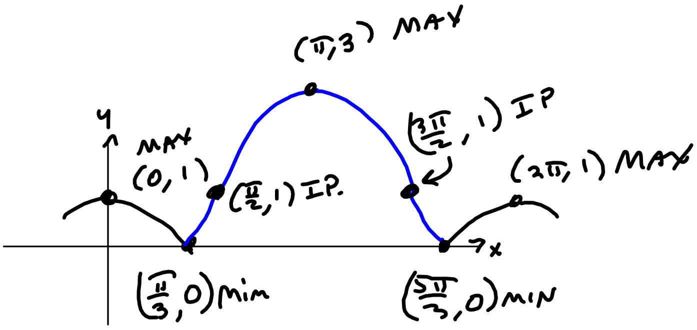
$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$A = (\frac{\pi}{3}, 0)$
$B = (\frac{5\pi}{3}, 0)$
(Need for c)

b. Ch 4 Evaluate $\int_0^{2\pi} f(x)\,dx$. $= \int_0^{2\pi} (2\cos(x) - 1)\,dx = \left[2\sin(x) - x\right]_0^{2\pi}$

$= 2\sin(2\pi) - 2\pi - [2\sin(0) - 0] = \boxed{-2\pi}$

c. Ch 4 Sketch a complete graph of $g(x) = |2\cos(x) - 1| = |f(x)|$ on the interval $[0, 2\pi]$.



$$|2\cos(x) - 1| = \begin{cases} 2\cos(x) - 1 & \text{if } 0 \le x \le \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \le x \le 2\pi \\ -(2\cos(x) - 1) & \text{if } \frac{\pi}{3} < x < \frac{5\pi}{3} \end{cases}$$

d. Ch 4 Evaluate $\int_0^{2\pi} g(x)\,dx$.

$$\Rightarrow \int_0^{2\pi} |2\cos(x)-1|\,dx = \int_0^{\frac{\pi}{3}} (2\cos(x)-1)\,dx - \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos(x)-1)\,dx + \int_{\frac{5\pi}{3}}^{2\pi} (2\cos(x)-1)\,dx$$

$$= 2\int_0^{\frac{\pi}{3}} (2\cos(x)-1)\,dx - \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos(x)-1)\,dx$$

$$= 2\left[2\sin(x)-x\right]_0^{\frac{\pi}{3}} - \left[2\sin(x)-x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$= 2\left[2\sin\left(\tfrac{\pi}{3}\right)-\tfrac{\pi}{3}\right] - \left[2\sin\left(\tfrac{5\pi}{3}\right) - \tfrac{5\pi}{3} - \left(2\sin\left(\tfrac{\pi}{3}\right) - \tfrac{\pi}{3}\right)\right]$$

$$= 4\left(\tfrac{\sqrt{3}}{2}\right) - \tfrac{2\pi}{3} - \left[2\left(-\tfrac{\sqrt{3}}{2}\right) - \tfrac{5\pi}{3} - \left(2\cdot\tfrac{\sqrt{3}}{2} - \tfrac{\pi}{3}\right)\right]$$

$$= 2\sqrt{3} - \tfrac{2\pi}{3} + \sqrt{3} + \tfrac{5\pi}{3} + \sqrt{3} - \tfrac{\pi}{3}$$

$$= \boxed{4\sqrt{3} + \tfrac{2\pi}{3}}$$

4. Evaluate the indefinite integrals.

a. Ch 4 (5 pts) $\int (2x-3)^4\,dx$ $= \frac{1}{2}\int u^4\,du$, where $u = 2x-3$ & $du = 2dx$

$= \frac{1}{2}\frac{u^5}{5} + C = \boxed{\frac{(2x-3)^5}{10} + C}$

$u = 2x-3 \Rightarrow du = 2dx \rightarrow dx = \frac{du}{2}$

b. Ch 4 (5 pts) $\int (2x-3)^4 x^2\,dx$ $\quad \frac{u+3}{2} = x$

$= \frac{1}{2}\int u^4\left(\frac{u+3}{2}\right)^2 du = \frac{1}{2}\left(\frac{1}{4}\right)\int u^4(u^2+6x+9)\,du$

$= \frac{1}{8}\int (u^6+6u^5+9u^4)\,du = \frac{1}{8}\left[\frac{(2x-3)^7}{7} + (2x-3)^6 + \frac{9(2x-3)^5}{5}\right] + C$

$= \boxed{\frac{(2x-3)^7}{56} + \frac{(2x-3)^6}{8} + \frac{9(2x-3)^5}{40} + C}$

c. Ch 4 (5 pts) $\int \sec^4(x)\tan(x)\,dx$ $= \int \sec^3(x)\left(\sec(x)\tan(x)\right)dx$

$= \int u^3\,du = \frac{u^4}{4} + C = \boxed{\frac{\sec^4(x)}{4} + C}$

d. Ch 6 & Ch 4 $\int \sin(x)\cdot e^{\cos(x)}\,dx$ $= -\int \left(e^{\cos(x)}\right)\left(-\sin(x)\,dx\right)$

$= -\int e^u\,du = -e^u + C = \boxed{-e^{\cos(x)} + C}$

5. Ch 4 Suppose I'm pacing back and forth, thinking my usual deep thoughts, and my rate of speed is given by $r(t)$, in feet per second. Tell me what the following integrals represent:

a. (5 pts) $\int_0^{3600} |r(t)|\, dt$ = TOTAL DISTANCE TRAVELED IN 1 HOUR, IN FEET

b. (5 pts) $\int_0^{3600} r(t)\, dt$ = NET CHANGE IN MY POSITION.

6. Perform the indicated differentiation:

a. Ch 4 (5 pts) $\frac{d}{dx}\int_0^{x} \frac{\sin(3t)}{t^2+4}dt = \boxed{\frac{\sin(3x)}{x^2+4}}$

b. Ch 4 (5 pts) $\frac{d}{dx}\int_{x^2}^{\cos(x)} \frac{\sin(3t)}{t^2+4}dt = \frac{d}{dx}\left[-\int_0^{x^2} \frac{\sin(3t)}{t^2+4}dt + \int_0^{\cos(x)} \frac{\sin(3t)}{t^2+4}dt\right]$

$$= \boxed{-\left(\frac{\sin(3x^2)}{x^4+4}\right)(2x) + \left(\frac{\sin(3\cos(x))}{9\cos^2(x)+4}\right)(-\sin(x))}$$

7. The function $f(x) = x^2 - 3x + 11$ is 1-to-1 on the restricted domain $D = \left[\frac{3}{2}, \infty\right)$.

a. Ch 6 (10 pts) Find the inverse function $f^{-1}(x)$. State its domain and range.

$$x^2 - 3x + 11 = y$$
$$x^2 - 3x + \left(\tfrac{3}{2}\right)^2 = y - 11 + \tfrac{9}{4}$$
$$\left(x - \tfrac{3}{2}\right)^2 = y - \tfrac{35}{4}$$
$$x - \tfrac{3}{2} = \pm\sqrt{y - \tfrac{35}{4}}$$
$$x = \tfrac{3}{2} \pm \sqrt{y - \tfrac{35}{4}}$$

We want "+"

$$f^{-1}(x) = \tfrac{3}{2} + \sqrt{x - \tfrac{35}{4}}$$
$$\mathcal{D} = \left[\tfrac{35}{4}, \infty\right)$$
$$\mathcal{R} = \left[\tfrac{3}{2}, \infty\right)$$

$$x^2 - 3x + 11$$
$$= x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 11$$
$$= \left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + \tfrac{44}{4}$$
$$= \left(x - \tfrac{3}{2}\right)^2 + \tfrac{35}{4} \quad (h,k) = \left(\tfrac{3}{2}, \tfrac{35}{4}\right)$$
$$\mathcal{D} = \left[\tfrac{3}{2}, \infty\right)$$
$$\mathcal{R} = \left[\tfrac{35}{4}, \infty\right)$$

b. Ch 6 (5 pts) Find $\left(f^{-1}\right)'(5)$, directly, by differentiating your answer for part a.

$$\left(f^{-1}\right)'(x) = \frac{d}{dx}\left[\tfrac{3}{2} + \sqrt{x - \tfrac{35}{4}}\right] = \frac{d}{dx}\left[\tfrac{3}{2} + \left(x - \tfrac{35}{4}\right)^{\frac{1}{2}}\right]$$
$$= \tfrac{1}{2}\left(x - \tfrac{35}{4}\right)^{-\frac{1}{2}} \longrightarrow \left(f^{-1}\right)'(5) = \tfrac{1}{2}\left(5 - \tfrac{35}{4}\right)^{-\frac{1}{2}} \quad \nexists$$

c. Ch 6 (5 pts) Find $\left(f^{-1}\right)'(5)$ by applying a theorem regarding derivatives of inverse functions.

See part d! Poorly-posed question!

8. (5 pts each) Find the derivative with respect to x.

a. Ch 6 $y = 7 \cdot 5^{x^2-3x}$ $\Rightarrow$ $y' = \left(7\ln(5) \cdot 5^{x^2-3x}\right)(2x-3)$

b. Ch 6 $y = \ln\left(\frac{\sqrt[5]{x^2-3x}}{\left(3x^5+5x\right)^3}\right)$ $= \frac{1}{5}\ln(x^2-3x) - 3\ln(3x^5+5x)$

$\Rightarrow$ $y' = \frac{1}{5}\left(\frac{2x-3}{x^2-3x}\right) - 3\left(\frac{15x^4+5}{3x^5+5x}\right)$

c. Ch 6 $y = \log_5\left(x^2-3x\right)$

$\Rightarrow$ $y' = \frac{2x-3}{\ln(5)(x^2-3x)}$

d. Ch 6 $y = \left[\cos(x)\right]^{x^2-3x}$ $\Rightarrow$ $\ln(y) = (x^2-3x)\ln(\cos(x))$

$\Rightarrow$ $\frac{y'}{y} = (2x-3)\ln(x^2-3x) + (x^2-3x)\left(\frac{-\sin(x)}{\cos(x)}\right)$

$\Rightarrow$ $y' = \left((2x-3)\ln(x^2-3x) - (x^2-3x)\tan(x)\right)\left(\cos(x)^{x^2-3x}\right)$

9. Ch 6 (10 pts) The half-life of Millsium is 75 years. How old is a Mills skeleton from a burial mound if there is 17% of its natural radioactive Millsium remaining?

$N(t) = N_0 e^{kt}$

$\frac{1}{2}$-life is 75 yrs :

$N_0 e^{75k} = \frac{1}{2} N_0$

$e^{75k} = \frac{1}{2}$

$75k = \ln(\frac{1}{2}) = -\ln(2)$

$k = -\frac{\ln(2)}{75}$

17% remains : $N_0 e^{kt} = .17 N_0$

$e^{kt} = .17$

$kt = \ln(.17)$

$t = \frac{\ln(.17)}{k} = \frac{75\ln(.17)}{-\ln(2)} \approx$

$\approx$ 191.7295012 years old

check

0	100
75	50
150	25
225	12.5

17% is here (between 25 and 12.5)

between 150 & 225 yrs ✓