FTC I

Let $g(x) = \int_0^x \frac{\sin(t)}{t^2+t+1}\,dt$

$$g'(x) = \frac{\sin(x)}{x^2+x+1}$$

what about $g'(x^2+2x)$? $\left(\frac{dg}{d(x^2+2x)}\right)\left(\frac{d(x^2+2x)}{dx}\right)$

$$= \frac{d}{dx}\int_0^{x^2+2x} \frac{\sin(t)}{t^2+t+1}\,dt = \left(\frac{\sin(x^2+2x)}{(x^2+2x)^2+(x^2+2x)+1}\right)(2x+2)$$

$$\int_a^b f = \int_a^0 f + \int_0^b f = -\int_0^a f + \int_0^b f$$

$$\frac{d}{dx}\int_{\sin(x)}^{\frac{x^2}{3}} \frac{t^2-5t}{\sin^2(t)}\,dt = \frac{d}{dx}\left[\int_{\sin(x)}^0 \sim + \int_0^{\frac{x^2}{3}} \sim\right]$$

$$= \frac{d}{dx}\left[-\int_0^{\sin(x)} \frac{t^2-5t}{\sin^2(t)}\,dt + \int_0^{\frac{x^2}{3}} \frac{t^2-5t}{\sin^2(t)}\,dt\right]$$

$$= -\left(\frac{\sin^2(x)-5\sin(x)}{\sin^2(\sin(x))}\right)(\cos(x)) + \left(\frac{\left(\frac{x^2}{3}\right)^2-5\left(\frac{x^2}{3}\right)}{\sin^2\left(\frac{x^2}{3}\right)}\right)\left(\frac{2}{3}x\right)$$

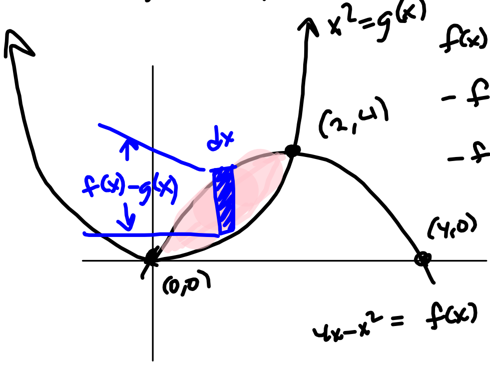
I want to have in-person final open Monday and Tuesday next week.

Part 1: Chapters 1 - 4 - 3 hrs

Part II: Chapters 5, 6 - 3 hrs?

Find the area bdd by

5.1 # 14 $y = x^2$, $y = 4x - x^2 = x(4-x)$



$f(x) = -x^2 + 4x$

$-f(x) = x^2 - 4x$

$-f(x) + 4 = x^2 - 4x + 2^2$

$-f(x) + 4 = (x-2)^2$

$-f(x) = (x-2)^2 - 4$

$f(x) = -(x-2)^2 + 4$

$(h,k) = (2,4)$

$y = g(x) = x^2$

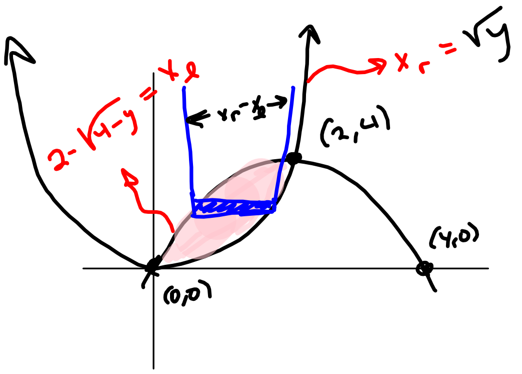
$g(2) = 4$

$$\text{Area} = \int_0^2 (y_u - y_l)\,dx$$

$$= \int_0^2 (f(x) - g(x))\,dx = \int_0^2 (4x - x^2 - x^2)\,dx = \int_0^2 (4x - 2x^2)\,dx$$

$$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 2(2)^2 - \frac{2}{3}(2)^3 - (0 - 0)$$

$$= 8 - \frac{16}{3} = \frac{24 - 16}{3} = \boxed{\frac{8}{3} = \text{Area}}$$



$$y = 4x - x^2 = -(x-2)^2 + 4 = y$$

Solve for x:

$$-(x-2)^2 = -4 + y$$
$$(x-2)^2 = 4 - y$$
$$x - 2 = \pm\sqrt{4-y}$$
$$x = 2 \pm \sqrt{4-y}$$
$$x_\ell = x = 2 - \sqrt{4-y} = h(y)$$

$$y = x^2 \Rightarrow x = \pm\sqrt{y}$$
$$x = \sqrt{y} = K(y)$$

$$\text{Area} = \int_0^4 (x_r - x_\ell)\,dx$$
$$= \int_0^4 \left(\sqrt{y} - \left(2 - \sqrt{4-y}\right)\right) dy = \int_0^4 \left(y^{\frac{1}{2}} - 2 + \underbrace{(4-y)}_{u=4-y}{}^{\frac{1}{2}}\right) dy$$

$u = 4-y$
$du = -dy$
$dy = -du$

$$= \left[\tfrac{2}{3}y^{\frac{3}{2}} - 2y\right]_0^4 - \int_0^4 (4-y)^{\frac{1}{2}}(-dy)$$

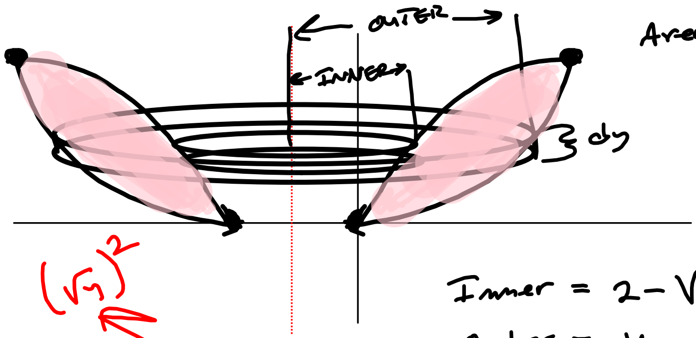
$$\int (4-y)^{\frac{1}{2}}\,dy$$
$$= \int u^{\frac{1}{2}}(-du)$$

$$= \tfrac{2}{3}(4)^{\frac{3}{2}} - 2(4) - (0-0) - \left[\tfrac{2}{3}(4-y)^{\frac{3}{2}}\right]_0^4$$

$$= \tfrac{2}{3}(2)^3 - 8 - \left[\tfrac{2}{3}(0) - \tfrac{2}{3}(4)^{\frac{3}{2}}\right]$$

$$= \tfrac{16}{3} - \tfrac{24}{3} + \tfrac{2}{3}(2)^3 = -\tfrac{8}{3} + \tfrac{16}{3} = \boxed{\tfrac{8}{3} = \text{Area}}\,!$$

Revolve that region about x = - 3



$$\text{Area} = \pi \int_0^4 (\text{OUTER}^2 - \text{INNER}^2)\, dy$$

$$(\sqrt{y})^2$$

$$\text{Inner} = 2 - \sqrt{4-y} = x_i$$

$$\text{Outer} = y = x_o$$

$$\text{So volume} = \boxed{\pi \int_0^4 \left(y^2 - (2-\sqrt{4-y})^2\right) dy} \rightarrow \text{STOP HERE ON WRITTEN TEST}$$

BONUS (come back, if time)

$$= \pi \int_0^4 y^2 - (4 - 4\sqrt{4-y} + 4 - y)\,dy$$

$$= \pi \int_0^4 (y^2 - 4 + 4\sqrt{4-y} - 4 + y)\,dy$$

$$= \pi \int_0^4 (y^2 + 4(4-y)^{\frac{1}{2}} + y - 8)\,dy$$

$$= \pi \left[\tfrac{1}{3}y^3 - \frac{4(4-y)^{\frac{3}{2}}}{\frac{3}{2}} + \tfrac{1}{2}y^2 - 8y \right]_0^4$$

$$= \pi \left[\tfrac{1}{3}y^3 - \tfrac{8}{3}(4-y)^{\frac{3}{2}} + \tfrac{1}{2}y^2 - 8y \right]_0^4$$

$\frac{4^2}{2} = 8$

$$= \pi \left[\tfrac{1}{3}(4)^3 - \tfrac{8}{3}(0)^{\frac{3}{2}} + \tfrac{1}{2}(16) - 32 - \left(-\tfrac{8}{3}(4)^{\frac{3}{2}}\right) \right]$$

$$= \pi \left[\tfrac{64}{3} + 8 - 32 + \tfrac{8}{3}(8) \right] = \pi \left[\tfrac{64}{3} + \tfrac{64}{3} - 24 \right]$$

8

$$\cancel{= \pi \left[\tfrac{128}{3} - \tfrac{72}{3} \right] = \pi \left[\tfrac{56}{3} \right]}$$

$$= \boxed{\frac{56\pi}{3} = \text{volume}}$$

$\left(8 - 24 + \frac{64}{3}\right)\pi$

$= \left(-16 + \frac{64}{3}\right)\pi$

$= \pi \cdot \frac{-48+64}{3} = \frac{16}{3}\pi$

consider:

$$(uv)' = u'v + uv'$$

$$\frac{d}{dx}[uv] = \left(\frac{d}{dx}u\right)v + u\frac{dv}{dx}$$

$$\int \frac{d}{dx}[uv]\,dx = \int \frac{du}{\cancel{dx}}\cdot v\,\cancel{dx} + \int u\frac{dv}{\cancel{dx}}\,\cancel{dx}$$

$$uv = \int v\,du + \int u\,dv \longrightarrow$$

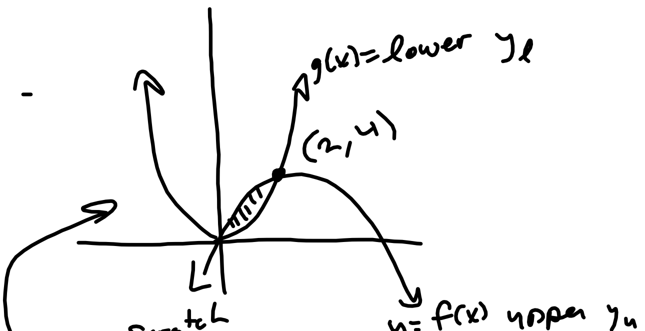
$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by Parts FORMULA.

Find the area bdd by

5.1 # 14 $y = x^2,\ y = 4x - x^2 = x(4-x)$

$$\text{Area} = \int (\text{upper} - \text{lower})\,dx = \int (g(x) - f(x))\,dx$$



Scratch

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2(x^2 - 2x) = 0$

$x(x-2) = 0$

$x \in \{0, 2\}$

$$= \int (y_u - y_l)\,dx$$

$$= \int_0^2 (4x - x^2 - x^2)\,dx$$

$$= \int_0^2 (4x - 2x^2)\,dx$$

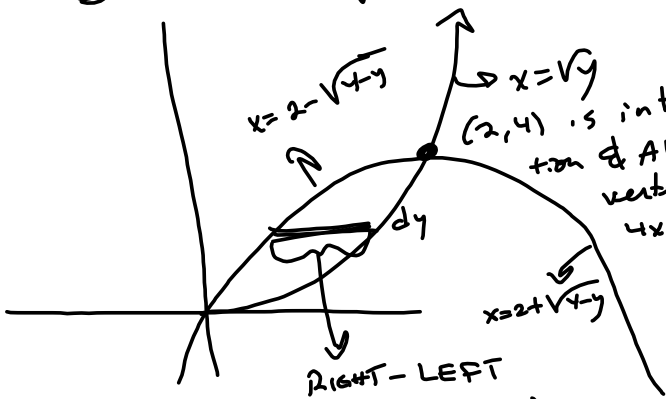
$$= \left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2$$

$$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$$

$$= 2(2)^2 - \frac{2}{3}(2)^3 - [0 - 0]$$

$$= 8 - \frac{16}{3} = \frac{24 - 16}{3} = \boxed{\frac{8}{3} = \text{Area}}$$

Do it sideways



(2,4) is intersection & Also the vertex of $4x-x^2$.

RIGHT − LEFT

$= \sqrt{y} - (2-\sqrt{4-y})$

$= y^{\frac{1}{2}} - 2 + (4-y)^{\frac{1}{2}}$

$$\text{Area} = \int_0^4 \left(y^{\frac{1}{2}} - 2 + (4-y)^{\frac{1}{2}}\right) dy$$

$$= \left[\tfrac{2}{3}y^{\frac{3}{2}} - 2y - \tfrac{2}{3}(4-y)^{\frac{3}{2}}\right]_0^4 =$$

$$\int (4-y)^{\frac{1}{2}} dy = -\int (4-y)^{\frac{1}{2}} dy$$

$$= -\tfrac{2}{3}(4-y)^{\frac{3}{2}} + C$$

$$= \tfrac{2}{3}(4)^{\frac{3}{2}} - 2(4) - \tfrac{2}{3}(0)^{\frac{3}{2}} - \left[\tfrac{2}{3}(0)^{\frac{3}{2}} - 2(0) - \tfrac{2}{3}(4)^{\frac{3}{2}}\right]$$

$$= \tfrac{2}{3}(2)^3 - 8 + \tfrac{2}{3}(2)^3 = \tfrac{16}{3} + \tfrac{16}{3} - \tfrac{24}{3} = \frac{32-24}{3} = \boxed{\tfrac{8}{3} = \text{Area}}$$

Scratch

$y = x^2 \longrightarrow x = \pm\sqrt{y} \longrightarrow$

$x = +\sqrt{y}$ by pic.

$y = -x^2 + 4x$

$-x^2 + 4x = y$

$-(x^2 - 4x \quad) = y$

$-(x^2 - 4x + 2^2) = y - 4$

$-(x-2)^2 = y - 4$

$(x-2)^2 = 4 - y$

$x - 2 = \pm\sqrt{4-y}$

$x = 2 \pm \sqrt{4-y}$

I'd like to do the area bounded by x = 0, y = sin(x) and y = cos(x) but we don't know our inverse trig functions yet, and the derivation requires some serious theory work:

To fully replicate the work done on WP#2, where you find the area two ways:

by integrating with respect to x and by integrating wrt y.

We could do it, but we'd have to know the antiderivatives of arcsin(y) and arccos(y).

To derive that formula requires integration by parts, which is also theory we don't yet have.

I propose a quick primer on integration by parts.

$$\int u\,dv = uv - \int v\,du$$

Consider $(uv)' = u'v + uv'$

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}.$$

$$\Longrightarrow \int \frac{d}{dx}[uv]\,dx = \int \frac{du}{dx} v\,dx + \int u \frac{dv}{dx}\,dx$$

$$\Longrightarrow uv = \int v\,du + \int u\,dv$$

$$\Longrightarrow \boxed{\int u\,dv = uv - \int v\,du} \; !$$

THE ALL-PURPOSE INTEGRATION TOOL / LEVER

We will use this to obtain $\int \arccos(x)\,dx$ & $\int \arcsin(x)\,dx$ & get a head start on Calc II!

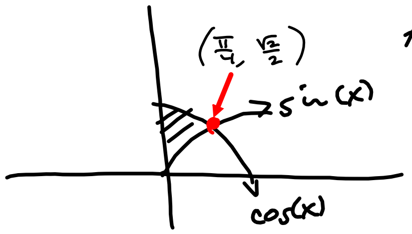
FOR TOMORROW

$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

$$\int \arccos(x)\,dx = x\cos(x) - \sqrt{1-x^2} + C$$

Area Between Two Curves.

Area bdd by $f(x) = \sin(x)$, $g(x) = \cos(x)$, $x = 0$



$$\text{Area} = \int_a^b (\text{upper} - \text{lower})\,dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x))\,dx$$

$$= \Big[\sin(x) - (-\cos(x))\Big]_0^{\frac{\pi}{4}}$$

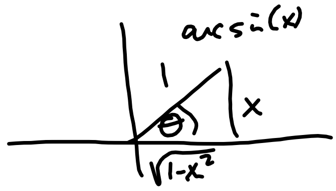
$$= \sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} - [\sin(0) + \cos(0)] = \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} - [0+1] = \boxed{\sqrt{2} - 1 = \text{Area}}$$

To do it wrt y, we need to learn

$\sin^{-1}(x) = \arcsin(x) =$ INVERSE SINE

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin(x))}$$

$$\boxed{\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \end{aligned}}$$



$$= \frac{1}{\cos\theta}$$

$$\cos\theta = \sqrt{1-x^2} \longrightarrow$$

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

$=$

$\int \arcsin(x)\,dx$

$u = \arcsin(x) \rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$ &) scratch

$dv = dx \qquad v = x$

$= uv - \int v\,du = (\arcsin(x))x - \int x\left(\frac{1}{\sqrt{1-x^2}}\,dx\right) = x\arcsin(x) - I.$

Now, $I = \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x\,dx)$

$= -\frac{1}{2}\left(\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right) + \hat{C} = -\sqrt{1-x^2} + \hat{C}$

Therefore, $\int \arcsin(x)\,dx = x\arcsin(x) - I$

$= x\arcsin(x) - (-\sqrt{1-x^2} + \hat{C}) = x\arcsin(x) + \sqrt{1-x^2} + C,$

where $C = -\hat{C}$ (to make it pretty).

We also need $\int \arccos(x)\,dx$

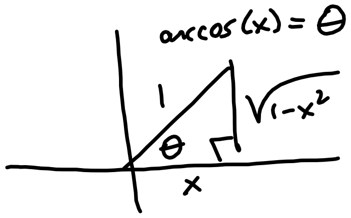
$g(x) = \cos(x)$

$g^{-1}(x) = \arccos(x) = \underset{\text{Special}}{\cos^{-1}}(x)$

$g'(x) = -\sin(x)$

$\frac{d}{dx}[\arccos(x)] = \frac{d}{dx}[g^{-1}(x)] = \frac{1}{g'(g^{-1}(x))}$

$= \frac{1}{-\sin(\arccos(x))} = \frac{1}{-\sin\theta}$



$$\Longrightarrow -\sin\theta = -\sqrt{1-x^2}$$

$$\Longrightarrow (g^{-1})'(x) = \frac{d}{dx}\left[\arccos(x)\right] =$$

$$= \frac{1}{-\sin\theta} \boxed{= \frac{1}{-\sqrt{1-x^2}} = \frac{d}{dx}\left[\arccos(x)\right]}$$

Now, derive $\int \arccos(x)\,dx$ by

Integrating by parts.

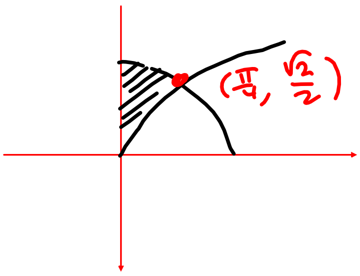
$$\left(\begin{array}{l} u = \arccos(x) \Longrightarrow du = -\frac{1}{\sqrt{1-x^2}}\,dx \\ dv = dx \longrightarrow v = x \end{array}\right) \longrightarrow$$

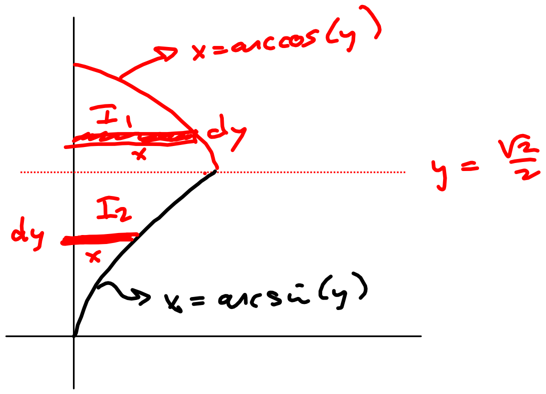
$$\int u\,dv = uv - \int v\,du = x\arccos(x) - \int x\left(-\frac{1}{\sqrt{1-x^2}}\,dx\right)$$

$$= x\arccos(x) - \frac{1}{2}\int \underbrace{(1-x^2)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}}\underbrace{(-2x\,dx)}_{du} = x\arccos(x) - \frac{1}{2}\left(\frac{\overset{\frac{u^{\frac{1}{2}}}{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}}{\frac{1}{2}}\right) + C$$

$$\boxed{\int \arccos(x)\,dx = x\arccos(x) - \sqrt{1-x^2} + C}$$

Now, we're ready to do this one wrt y





$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\frac{4\sqrt{2} - \sqrt{2}\pi}{8}$

Area $= I_1 + I_2$

$$I_1 = \int_{\frac{\sqrt{2}}{2}}^{1} \arccos(y)\,dy = \Big[y \arccos(y) - \sqrt{1-y^2}\Big]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \arccos(1) - \sqrt{1-(1)^2} - \left[\frac{\sqrt{2}}{2}\arccos\left(\frac{\sqrt{2}}{2}\right) - \sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}\right]$$

$$= 0 - 0 - \left[\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} - \sqrt{1-\frac{1}{2}}\right] = -\frac{\sqrt{2}\,\pi}{8} + \sqrt{\frac{1}{2}} =$$

$$I_2 = \int_0^{\frac{1}{\sqrt{2}}} \arcsin(y)\,dy$$

$$= \Big[y \arcsin(y) + \sqrt{1-y^2}\Big]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\arcsin\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$-\left[0 + \sqrt{1}\right] = \frac{1}{\sqrt{2}}\cdot\frac{\pi}{4} + \frac{1}{\sqrt{2}} - 1$$

$$\Longrightarrow I_1 + I_2 = \frac{-\sqrt{2}\,\pi}{8} + \frac{1}{\sqrt{2}} + \frac{\sqrt{2}\,\pi}{8} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2}{\sqrt{2}} - 1 = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$ = Area the $\int x\,dy$ way

$= \int f^{-1}(y)\,dy$