Fundamental Theorem of Calculus Part I (FTC I) - The derivative of the antiderivative is the integrand.

FTC I Let f be cnts on $[a,b]$, and suppose $x \in [a,b]$.

Then if

$$g(x) = \int_a^x f(t)\,dt \text{ for } x \in [a,b]. \text{ Then}$$

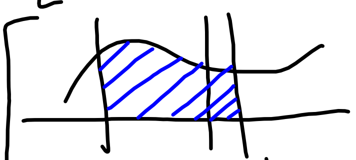
$$g'(x) = f(x).$$

Proof

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[g(x+h) - g(x)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$

$= \lim_{h \to 0} \left(\frac{1}{h}\right[$



(Assume $h>0$)

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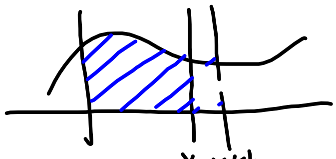
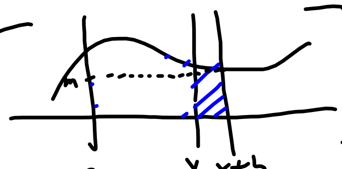


(Assume $h>0$)

$\left]\right)$

$= \lim_{h \to 0} \left(\frac{1}{h}\right[$



(Assume $h>0$)

$$\left]\right) = \lim_{h \to 0} \left[\frac{1}{h} \int_x^{x+h} f(t)\,dt\right]$$

By Extreme Value Theorem (EVT), $f(t)$ is cnt^{s} $\Longrightarrow$

$\exists\, u \in [x, x+h] \ni f(u) = \min\limits_{[x,x+h]} f(t)$ &

$\exists\, v \in [x, x+h] \ni f(v) = \max\limits_{[x,x+h]} f(t)$

$[a,b] = [x, x+h] \Longrightarrow$

$b-a = x+h-x = h$

Then $\quad f(u)\,h \le \lim\limits_{h\to 0} \int_x^{x+h} f(t)\,dt \le f(v)\,h$

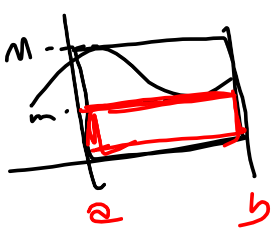
$m(b-a) \qquad\qquad M(b-a)$

Squeeze it! $\quad v \to x \quad h \to 0 \qquad\qquad h \to 0 \quad v \to x$

$f(x)\,h \le \lim\limits_{h\to 0} \int_x^{x+h} f(t)\,dt \le f(x)\,h$

$\Longrightarrow f(x) = \underbrace{\lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt}_{g'(x) \,=\, f(x)\,!} \le f(x)$

$\blacksquare$



Fundamental Theorem of Calculus Part II. The definite integral is the net change in *any* antiderivative.

FTC II Let $f(x)$ be cnts on $[a,b]$ and let $F(x)$ be <u>any</u> antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

<u>Proof</u> Define $g(x) = \int_a^x f(t)\,dt \quad \forall\, x \in [a,b]$.

Then $g(a) = \int_a^a f(t)\,dt = 0$

NOTE $g(x) = F(x) + C$, b/c
$F'(x) = f(x) = g'(x)$

and $g(b) = F(b) + C$
& $g(a) = F(a) + C$

$$g(b) - g(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$$

$$= \int_a^b f(t)\,dt = (F(b)+C) - (F(a)+C)$$

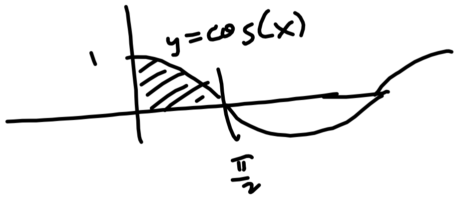
$$= F(b) - F(a) \; \blacksquare$$

Recall

$\frac{d}{dx}\sin(x) = \cos(x) \Rightarrow$

$\int \cos(x)\,dx = \sin(x) + C$

$$\int_0^{\frac{\pi}{2}} \cos(x)\,dx = \sin(x)\Big]_0^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = \boxed{1}$$

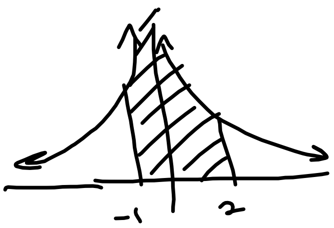


$= 1$

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \int_{-1}^{2} x^{-2}\,dx = \frac{x^{-1}}{-1}\Big]_{-1}^{2} = -\frac{1}{x}\Big]_{-1}^{2} = -\frac{1}{2} - \left(-\frac{1}{-1}\right)$$

$$= -\frac{1}{2} - \frac{2}{2} = -\frac{3}{2} \ ?!$$

Less than zero?
But $\frac{1}{x^2} > 0$ on its domain!



$\perp$

Classic NONEXAMPLE.

$\frac{1}{x^2}$ is not cnts on $[-1,2]$, because

$\frac{1}{0^2}$ $\nexists$