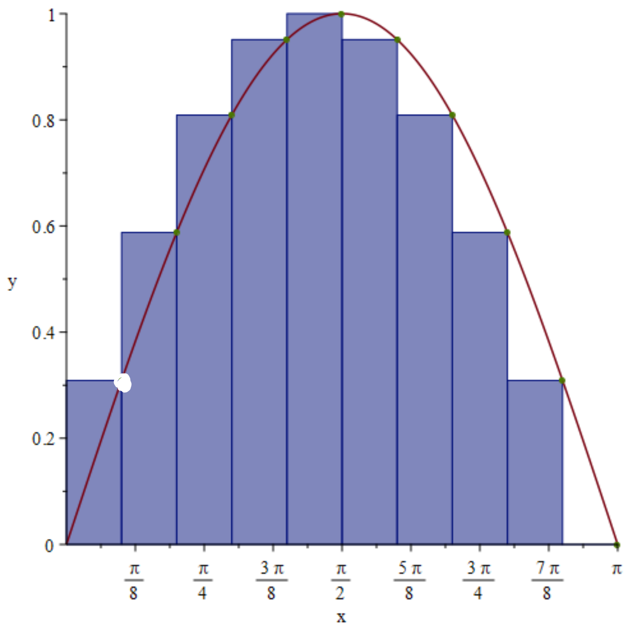


An approximatiion of the area under the sine curve, from 0 to 2Pi, using right endpoints and n = 10 approximating rectangles.

"Mesh" of the partition $= \Delta x =$ width of the rectangles

$= \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$ = width

height of right endpoints:

$a+\Delta x,\ a+2\Delta x, \ldots, a+k\Delta x, \ldots, a+10\Delta x$ $(a=0)$

$x_1 = \frac{\pi}{10}, x_2 = 2\left(\frac{\pi}{10}\right), x_3 = 3\left(\frac{\pi}{10}\right), \ldots, x_k = k\left(\frac{\pi}{10}\right) = \frac{\pi}{10}k = \frac{\pi k}{10}$

height of the k^{th} rectangle is $f(x_k) = f(a+k\Delta x)$ $= f\left(\frac{\pi k}{10}\right)$

Area under the curve $\approx$ sum of the areas of rectangles

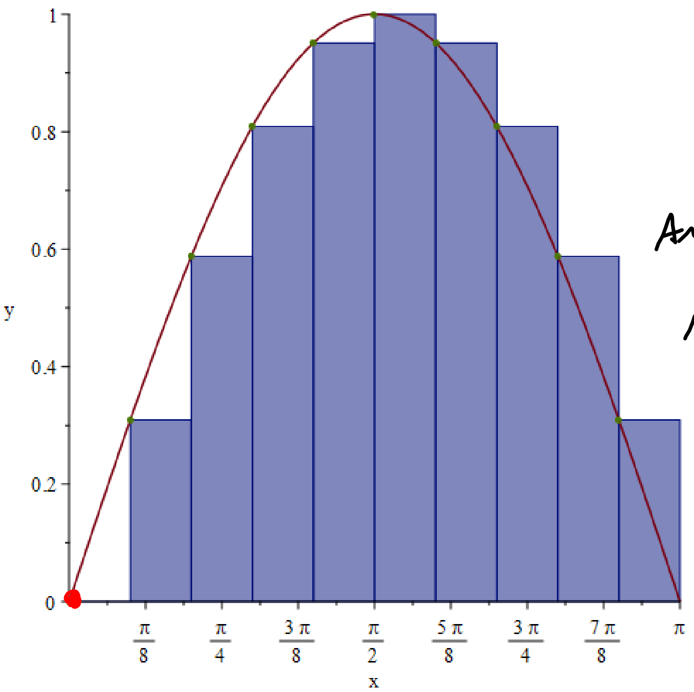
$= f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{10})\Delta x$

$= \left(f(x_1) + f(x_2) + \ldots + f(x_n)\right)\Delta x$

$= \left(\sum_{k=1}^{n} f(x_k)\right)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k)$

$= \frac{b-a}{n}\sum_{k=1}^{n} f(a+k\Delta x) = \frac{b-a}{n}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)$ $0+\frac{\pi}{10}k$

$= \frac{\pi}{10}\sum_{k=1}^{10}\sin\left(\frac{\pi}{10}k\right)$ is the Riemann Sum for RIGHT-ENDPOINT $\sin(x)$ on $[0,\pi]$



Left-endpoint formulation:

$k \longmapsto (k-1)$

$x_1 = 0,\ x_2 = \frac{\pi}{10},\ \ldots,\ x_{10} = \frac{\pi}{10} \cdot 9$

$$\text{Area} \approx \frac{\pi}{10} \sum_{k=1}^{10} \sin\left(\frac{\pi}{10}(k-1)\right)$$

Midpoint:

$x_1 = a + \frac{1}{2}\Delta x$

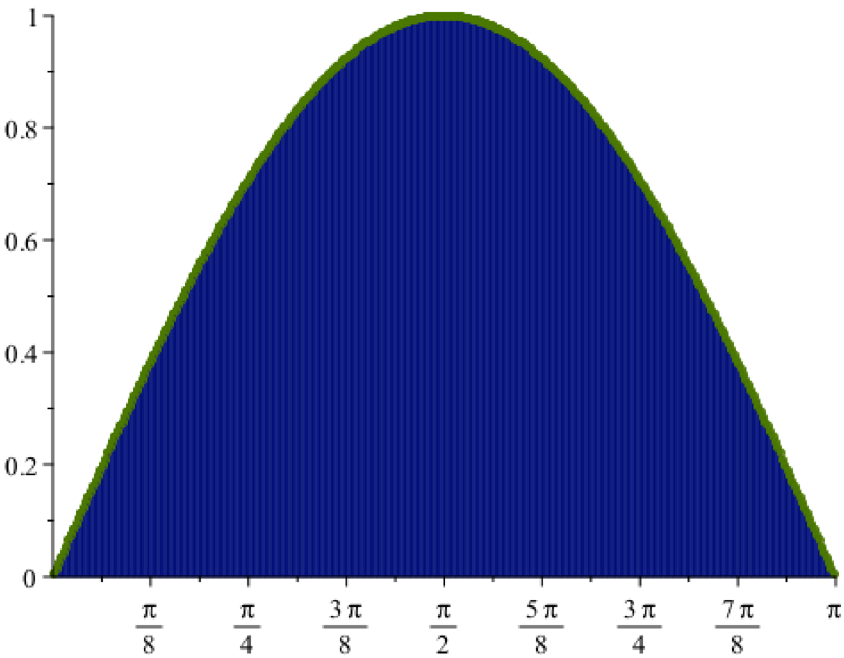
$x_2 = a + \frac{1}{2}\Delta x + \Delta x$

$= a + \frac{3}{2}\Delta x$

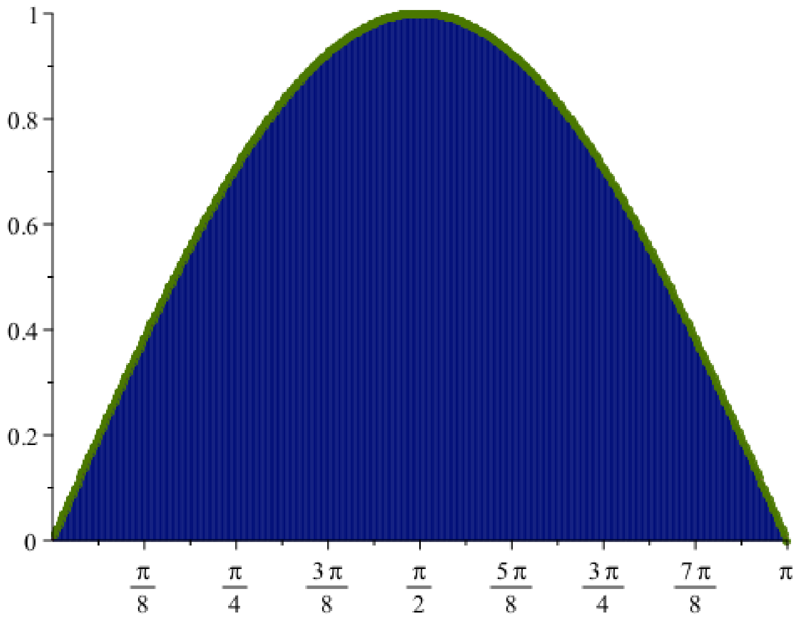
$x_3 = a + \frac{5}{2}\Delta x$

$\vdots$

$x_k = a + \frac{2k-1}{2}\Delta x$



An animated Riemann sum midpoint approximation of $\int_0^\pi f(x)\,dx$, where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 2.000008031. Number of subintervals used: 320.



An animated right Riemann sum approximation of $\int_0^{\pi} f(x)\,dx$, where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 1.999983935. Number of subintervals used: 320.

ACTUAL area under the curve is:

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

$$= \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{\pi}{n}k\right) = \int_0^{\pi} \sin(x)\,dx$$

In the sequel, we will see that this is $-\cos(\pi) - (-\cos(0))$, b/c cosine is the antiderivative of sine!

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

This can be done for ANY function that is continuous!

This can be extended to any "measurable" function, with the Lebesgue Integral,

which is a generalization of what we're doing, and what we're doing is a generalization of the area of a rectangle! (and theory of limits)

$1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$ (Gauss @ age 6)

$\sum_{k=1}^{100} k = \frac{100(101)}{2} = 50(101) = \boxed{5050}$

$1+2+3+4+\cdots+96+97+98+99+100$

$\frac{100}{2}\cdot 101 = 5050$!

Proof by Induction

Principle of Mathematical Induction.

Show that the result holds for n = 1. If the case for n = k implies the case for n = k + 1, and you, then it is true for every n = 1, 2, 3, ...

Claim: If $n \in \mathbb{N}$, then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof $n=1 \Rightarrow \sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓

Assume true for some $1 \leq n \in \mathbb{N}$ & that we know

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then $\sum_{k=1}^{n+1} k = 1+2+\cdots+n+(n+1)$

Induction Hypothesis

$= \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2(n+1)}{2} = \frac{n^2+n+2n+2}{2}$

$= \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2} \Rightarrow$

Done, by PMI !

Why torture you like this? We want to find the area under a polynomial using the limit of a Riemann Sum (Old-School)

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + \text{lower degree}}{6} = \frac{n^3 + \text{lower}}{3}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{n^2 + \text{lower}}{2}\right]^2 = \frac{n^4 + \text{lower}}{4}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2 + \text{lower}}{2}$$

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ of 'em}} = n$$

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2} \qquad \text{FTC II}$$

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\left[\frac{n^2 + \text{lower}}{2}\right]$$

$$= \frac{n^2}{2n^2} + \frac{\text{lower degree}}{2n^2} \xrightarrow{n\to\infty} \frac{1}{2}$$

$\to 0$

$$x_k = a + \Delta x\, k = 0 + \frac{1}{n}k = \frac{k}{n}$$

$$f(x_k) = x_k$$