Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.

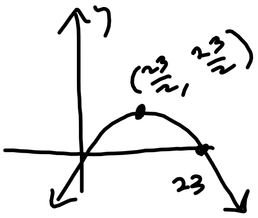
(a) Complete the table of values so that the sum of the numbers in the first two columns is always 23.

First number	Second number	Product
1	22	22
2	21	42
3	20	
4	19	
5		
6		
7		
8		
9		
10		
11		

	A	B	C	D	E	F
1						
2						
3						
4	3.7 Optimization Problem.					
5	We maximize the product of two numbers whose sum is 23					
6			First #	Second #	Product	
7			23	0	0	
8			22	1	22	
9			21	2	42	
10			20	3	60	
11			19	4	76	
12			18	5	90	
13			17	6	102	
14			16	7	112	
15			15	8	120	
16			14	9	126	
17			13	10	130	
18			12	11	132	
19			11	12	132	
20			10	13	130	
21			9	14	126	
22			8	15	120	
23			7	16	112	
24			6	17	102	
25			5	18	90	
26			4	19	76	
27			3	20	60	
28			2	21	42	
29			1	22	22	
30			0	23	0	
31			-1	24	-24	
32			-2	25	-50	
33			-3	26	-78	
34			-4	27	-108	

	A	B	C
	First #	Second #	Product
7	23	=23-A7	=A7*B7
8	=A7-1	=23-A8	=A8*B8
9	=A8-1	=23-A9	=A9*B9
10	=A9-1	=23-A10	=A10*B10
11	=A10-1	=23-A11	=A11*B11
12	=A11-1	=23-A12	=A12*B12
	=A12-1	=23-A13	=A13*B13
	=A13-1	=23-A14	=A14*B14
	=A14-1	=23-A15	=A15*B15
	=A15-1	=23-A16	=A16*B16
	=A16-1	=23-A17	=A17*B17
	=A17-1	=23-A18	=A18*B18
	=A18-1	=23-A19	=A19*B19
	=A19-1	=23-A20	=A20*B20
	=A20-1	=23-A21	=A21*B21
	=A21-1	=23-A22	=A22*B22
	=A22-1	=23-A23	=A23*B23
	=A23-1	=23-A24	=A24*B24
	=A24-1	=23-A25	=A25*B25
	=A25-1	=23-A26	=A26*B26
	=A26-1	=23-A27	=A27*B27
	=A27-1	=23-A28	=A28*B28
	=A28-1	=23-A29	=A29*B29
	=A29-1	=23-A30	=A30*B30
	=A30-1	=23-A31	=A31*B31
	=A31-1	=23-A32	=A32*B32
	=A32-1	=23-A33	=A33*B33

On the basis of the evidence in the table, estimate the answer to the problem. (Enter your answers as a comma-separated list.)



$x = 1^{st}$ #, $y = 2^{nd}$ #

want $x+y=23$ & xy as big as possible

Auxiliary Equation:

$x+y=23$

$y = 23-x \Rightarrow xy = x(23-x)$

$(\frac{23}{2}, \frac{23}{2})$ is it.

(b) Use calculus to solve the problem. (Enter your answers as a comma-separated list.)

$x(23-x) = 23x - x^2 = \text{Area} \Rightarrow$

$\frac{dA}{dx} = 23 - 2x \overset{SET}{=} 0$

$\Rightarrow -2x = -23$

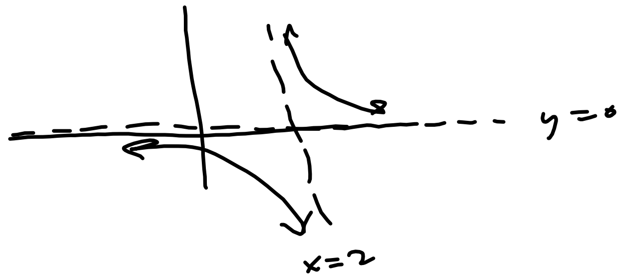
$\Rightarrow x = \frac{23}{2}$

$\Rightarrow y = 23 - \frac{23}{2} = \frac{46-23}{2} = \frac{23}{2}$

Infinite Limits and Limits at Infinity.

$\lim_{x \to 2} \frac{5}{x-2}$ $\nexists$

But for graphing, we can talk about infinite limits.



$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2^+} f(x) = +\infty$

Sign Pattern $\frac{5}{x-2}$

$-$ $+$

2

2 is a cut point.

(so are any zeros) $= 0$

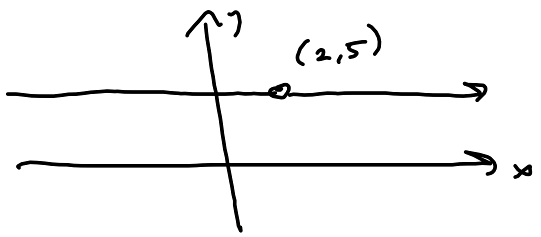
from $\lim_{x\to 2^-} f(x) = -\infty$

from $\lim_{x\to 2^+} f(x) = \infty$

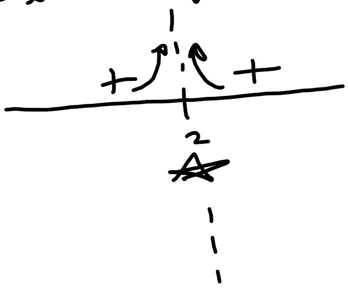
$x = 2$

$$\frac{5x-10}{x-2} = \frac{5\cancel{(x-2)}}{\cancel{x-2}} = 5 \quad x \neq 2$$

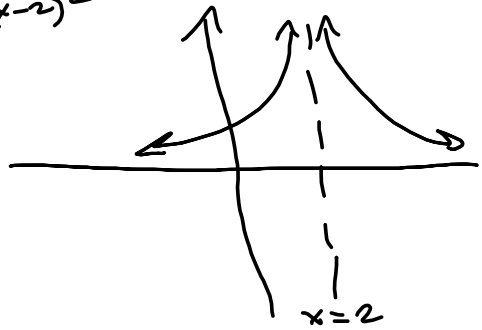
} Hole @ $x=2$



Sign Pattern



$$\lim_{x\to 3} \frac{1}{(x-2)^2} = \infty$$



$x=2$

Limits at infinity:

$\lim_{x \to -\infty} f(x)$ & $\lim_{x \to \infty} f(x)$: END BEHAVIOR

Rational Functions

$$\frac{p(x)}{g(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_2 x^2 + b_1 x + b_0}$$

$\deg(p(x)) = n$

$\deg(g(x)) = m$

1. Proper Rational Function

$m > n$ Horizontal Asymptote $\boxed{y = 0}$

$$\frac{325x^5 + 7x^4 - 11x}{.01x^6 + 5}$$

$m = 6 > 5 = n \Rightarrow$ $\boxed{y = 0 \text{ is H.A.}}$

$$\frac{x^5\left(325 + \frac{7}{x} - \frac{11}{x^4}\right)}{x^5\left(.01x + \frac{5}{x^5}\right)} \xrightarrow{x \to \pm\infty} \frac{325}{.01x} \xrightarrow{x \to \pm\infty} 0$$

② Improper Rational Function: $n \geq m$

ⓐ $n = m$: Divide Leading Coefficients

$$\lim_{x \to \pm\infty} \frac{5x^2 - 3x + 2}{7x^2 + 1x + 5} = \lim_{x \to \pm\infty} \frac{x^2\left(5 - \frac{3}{x} + \frac{2}{x^2}\right)}{x^2\left(7 - \frac{1}{x} + \frac{5}{x^2}\right)}$$

$$= \lim_{x \to \pm\infty} \frac{5 - \frac{3}{x} + \frac{2}{x^2}}{7 - \frac{1}{x} + \frac{5}{x^2}} = \boxed{\frac{5}{7} = y = H.A.}$$

$$\frac{5x^2}{7x^2} = \boxed{\frac{5}{7} = y = H.A.}$$

ⓑ $n > m$ You need to do some Polynomial Division

$$f(x) = 2x + 4 + \frac{1}{x-3} = \frac{(2x+4)(x-3) + 1}{x-3}$$

$$= \frac{2x^2 - 2x - 12 + 1}{x-3} = \frac{2x^2 - 2x - 11}{x-3}$$

$n = 2$
$m = 1$
$n > m$: Divide!

$$
\begin{array}{r}
2x+4 \quad r\,1 \\
x-3 \,\big)\, \overline{2x^2-2x-11} \\
-(2x^2-6x) \\
\hline
4x-11 \\
-(4x-12) \\
\hline
1
\end{array}
$$

$\frac{2x^2}{x} = 2x$

$\frac{4x}{x} = 4$

Synthetic:

$$
\begin{array}{r|rrr}
3 & 2 & -2 & -11 \\
 & & 6 & 12 \\
\hline
 & 2 & 4 & 1 \\
 & x^1 & x^0 & r
\end{array}
$$

This says $f(x) = 2x+4+\frac{1}{x-3}$

This means that

$f(x) = 2x+4+\frac{1}{x-3}$

$= \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$

$$
\begin{array}{r}
9 \; r\,1 \\
3 \,\big)\, \overline{28} \\
-27 \\
\hline
1
\end{array}
$$

means $28 = 3(9)+1$, i.e.

$\frac{28}{3} = 9 + \frac{1}{3}$

This says that as you get farther away from $x=3$, $f(x)$ acts more & more like $y=2x+4$, because

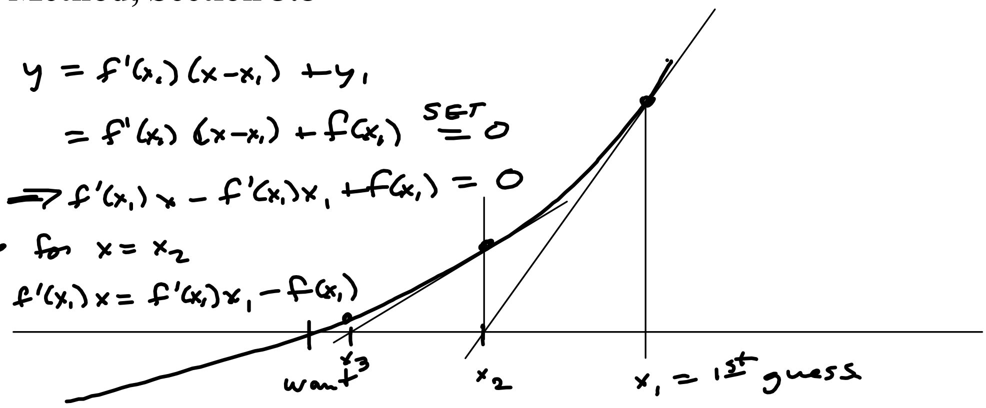
$\lim_{x\to\infty} \frac{1}{x-3} = 0$

I have college algebra resources and a writing project that covers a lot of the theory, here. Let me know if you want to see that stuff.

Here's a link: Click Here

Newton's Method, Section 3.8

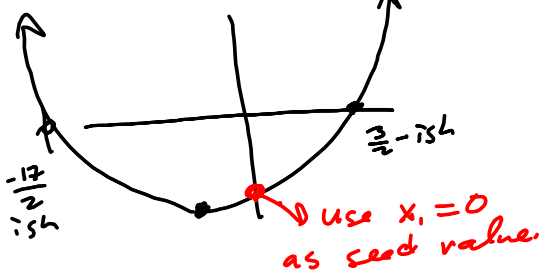
$$L(x) = y = f'(x_1)(x-x_1)+y_1$$
$$= f'(x_1)(x-x_1)+f(x_1) \overset{SET}{=} 0$$
$$\Rightarrow f'(x_1)x - f'(x_1)x_1 + f(x_1) = 0$$
Solve for $x = x_2$
$$f'(x_1)x = f'(x_1)x_1 - f(x_1)$$



Let's do this for $f(x) = x^2+7x-13$

$$x^2+7x+\left(\tfrac{7}{2}\right)^2 - \tfrac{49}{4} - \tfrac{52}{4}$$
$$= \left(x+\tfrac{7}{2}\right)^2 - \tfrac{101}{4} \qquad a(x-h)^2+k$$
$$(h,k) = \left(-\tfrac{7}{2}, -\tfrac{101}{4}\right) = (h,k)$$
$$\overset{SET}{=} 0 \Rightarrow x = \frac{-7 \pm \sqrt{101}}{2}$$
close to $\frac{-7\pm 10}{2}$ $\to -\frac{17}{2}$, $\frac{3}{2}$

$$L(x) = y = f'(x_1)(x-x_1)+y_1$$
$$= f'(x_1)(x-x_1)+f(x_1) \overset{SET}{=} 0$$
$$\Rightarrow f'(x_1)x - f'(x_1)x_1 + f(x_1) = 0$$
Solve for $x = x_2$
$$f'(x_1)x = f'(x_1)x_1 - f(x_1)$$
$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{OR}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{is cleaner}$$



We won't have you do many steps. I'm thinking of an Excel implementation as extra credit.

$f(x) = x^2 + 7x - 13$

$f'(x) = 2x + 7$

$x_1 = 0$ (1^{st} guess)

$f(x_1) = -13$

$f'(x_1) = 7$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{-13}{7} = \frac{13}{7}$

$x_3 = \frac{13}{7} - \frac{f(13/7)}{f'(13/7)}$ so it rapidly gets painful by hand!

$x_4 = \cdots$

$x_5 = \cdots$