3.1 Stuff that you already know. Amost everyone is done with 3.1

Page down to Page 13 to see 3.2 material.

I proved Rolle's and Mean Value Theorems in writing at the very end.

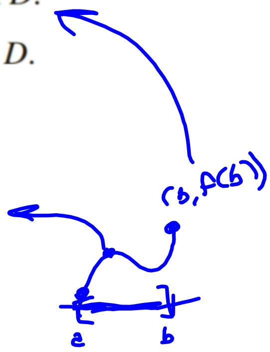
They're not that difficult, if you remember all the theorems that went before, but we didn't cover those formalities in class, because you guys are so fast.

$\boxed{1}$ **Definition** Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

$\boxed{2}$ **Definition** The number $f(c)$ is a

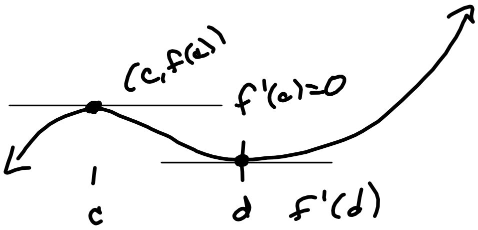
- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

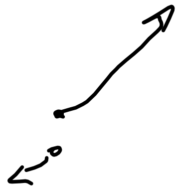


$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

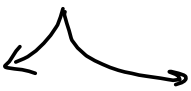
Set $f'=0$
solve.





6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

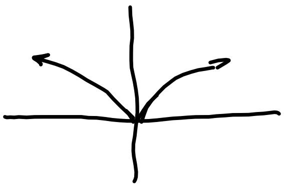
$-x^{2/3}$



$f'(0) \nexists$

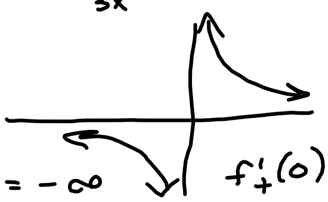
& $f(0)$ is a maximum.

$x^{2/3}$



$$y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{1/3}}$$

x^p for $0 < p < 1$



$f'_-(0) = -\infty$ $f'_+(0) = -\infty$

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Find where $f'=0$ & where f' DNE
on the domain of f.

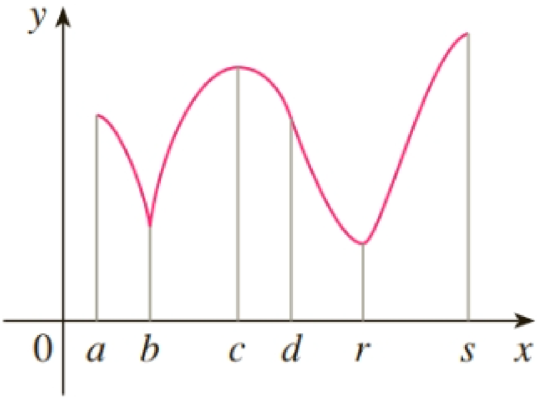
The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

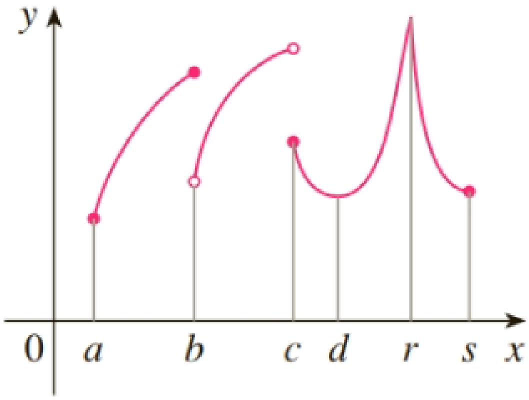
Summary: Check *f* at endpoints of its domain and at all critical values of *f*.

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.

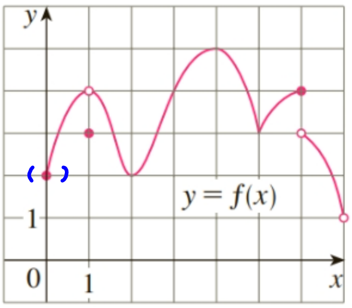


4.



5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

5.



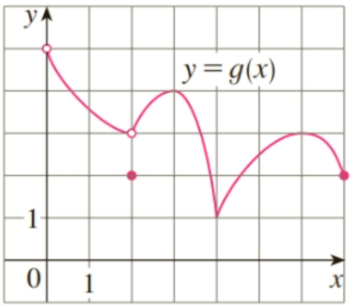
Global Min:
(0,2), (2,2)

Local min:
(2,2), (5,3)
(1,3)

Global Max:
(4,5)

Local Max
(4,5), (7,4)

6.



No Abs Max

Local Max
(3,4), (5,3)

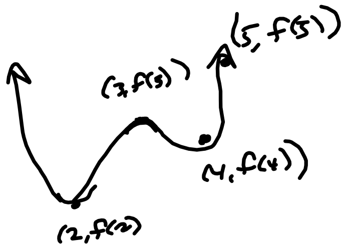
Local Min
(2,2) !
(4,1)

Abs Min
(4,1)

Not

7–10 Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

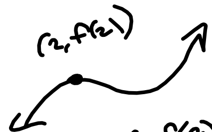


8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3

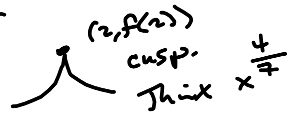
9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2

10. Absolute maximum at 2, absolute minimum at 5, 4 is a critical number but there is no local maximum or minimum there.

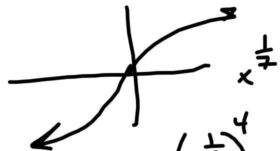
11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.

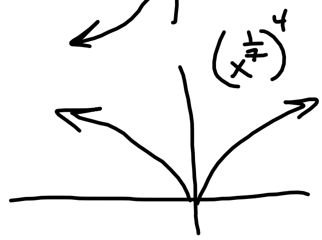


(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.



(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.





14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.

(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = \frac{1}{2}(3x - 1), \quad x \leqslant 3$

16. $f(x) = 2 - \frac{1}{3}x, \quad x \geqslant -2$

17. $f(x) = 1/x, \quad x \geqslant 1$

18. $f(x) = 1/x, \quad 1 < x < 3$

19. $f(x) = \sin x, \quad 0 \leqslant x < \pi/2$

20. $f(x) = \sin x, \quad 0 < x \leqslant \pi/2$

21. $f(x) = \sin x, \quad -\pi/2 \leqslant x \leqslant \pi/2$

22. $f(t) = \cos t, \quad -3\pi/2 \leqslant t \leqslant 3\pi/2$

23. $f(x) = 1 + (x + 1)^2, \quad -2 \leqslant x < 5$

24. $f(x) = |x|$

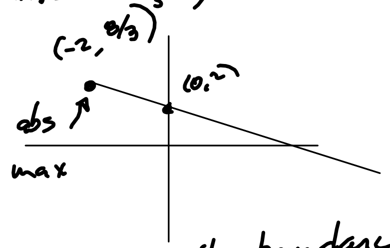
25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = 1 - x^3$

27. $f(x) = \begin{cases} x^2 & \text{if } -1 \leqslant x \leqslant 0 \\ 2 - 3x & \text{if } 0 < x \leqslant 1 \end{cases}$

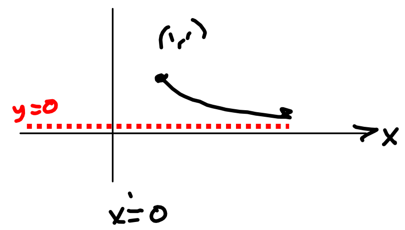
28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \leqslant x < 1 \\ 4 - 2x & \text{if } 1 \leqslant x \leqslant 3 \end{cases}$

#16 $f(x) = 2 - \frac{1}{3}x, \quad x \geq -2$

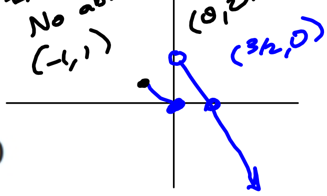


x=2 is its boundary

17 $f(x) = \frac{1}{x}$



27
No abs. max



29–42 Find the critical numbers of the function.

29. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

30. $f(x) = x^3 + 6x^2 - 15x$

31. $f(x) = 2x^3 - 3x^2 - 36x$

32. $f(x) = 2x^3 + x^2 + 2x$

33. $g(t) = t^4 + t^3 + t^2 + 1$

34. $g(t) = |3t - 4|$

32 $2x^3 + x^2 + 2x = f(x) \Rightarrow$

$f'(x) = 6x^2 + 2x + 2 \overset{\text{SET}}{=} 0$

$\Rightarrow 4x^2 + x + 1 = 0$

$b^2 - 4ac = 1^2 - 4(4)(1) = -15 < 0$

No real zeros

No critical ~~#s~~

x^p for $0 < p < 1$ is about the only situation where $f(c)$ exists but $f'(c)$ doesn't

$$x^{\frac{1}{2}} \rightsquigarrow \frac{1}{2x^{1/2}}$$

$$x^{\frac{1}{3}} \rightsquigarrow \frac{1}{3x^{-2/3}}$$

Usually, we just worry about

$$f' = 0$$

$$f' \nexists$$

f' can change signs

$$f'(x) = \frac{(x+2)(x-3)}{\left((x+5)(x-3)\right)^2}$$

45–56 Find the absolute maximum and absolute minimum values of f on the given interval.

45. $f(x) = 12 + 4x - x^2$, $[0, 5]$

46. $f(x) = 5 + 54x - 2x^3$, $[0, 4]$

47. $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

48. $f(x) = x^3 - 6x^2 + 5$, $[-3, 5]$

49. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, $[-2, 3]$

50. $f(t) = (t^2 - 4)^3$, $[-2, 3]$

51. $f(x) = x + \dfrac{1}{x}$, $[0.2, 4]$

52. $f(x) = \dfrac{x}{x^2 - x + 1}$, $[0, 3]$

53. $f(t) = t - \sqrt[3]{t}$, $[-1, 4]$

#46

$f(x) = -2x^3 + 54x + 5$ on $[0, 4]$

$f(0) = 5 \rightsquigarrow (0, 5)$

$$\begin{array}{r|rrrr} 4 & -2 & 0 & 54 & 5 \\ & & -8 & -32 & 88 \\ \hline & -2 & -8 & 22 & \boxed{93 = f(4)} \end{array} \rightsquigarrow (4, 93)$$

$f'(x) = -6x^2 + 54 \overset{\text{SET}}{=} 0$

$6x^2 = 54$

$x^2 = 9$

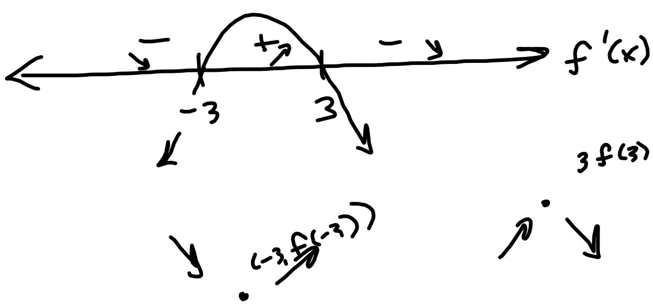
$x = \pm 3$

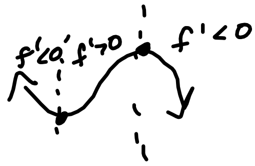
$$\begin{array}{r|rrrr} 3 & -2 & 0 & 54 & 5 \\ & & -6 & -18 & 108 \\ \hline & -2 & -6 & 36 & \boxed{113 = f(3)} \end{array}$$

$$\begin{array}{r|rrrr} -3 & -2 & 0 & 54 & 5 \\ & & 6 & -18 & -108 \\ \hline & -2 & 6 & 36 & \boxed{-103 = f(-3)} \end{array}$$

$(3, 113)$ local & abs

$(-3, -103)$ local & abs.





Sign change of the derivative!

Don't jump to the conclusion that there's a converse to Fermat's Theorem.

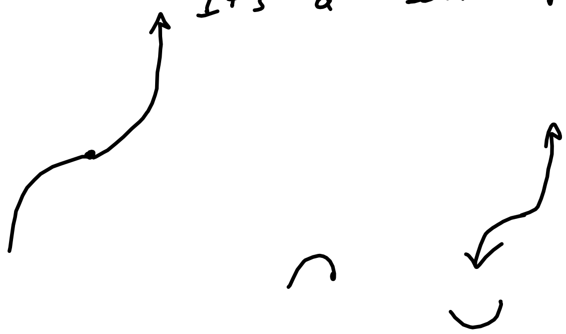
Just because the derivative is zero doesn't mean that you're looking at a max or a min:

$f(x) = x^3$

$f'(x) = 2x^2 = 0$ @ $x=0$, but

(0,0) isn't a max/min.

It's a "terrace point."

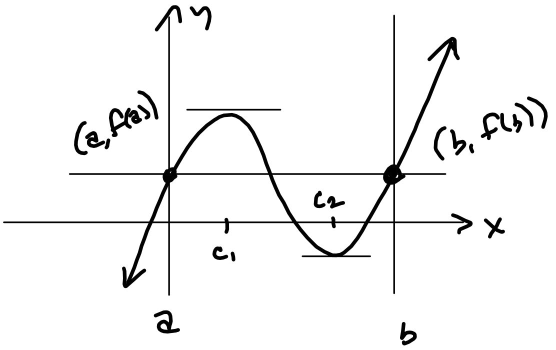


3.2 The Mean Value Theorem

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



I drew one with 2 such c's.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

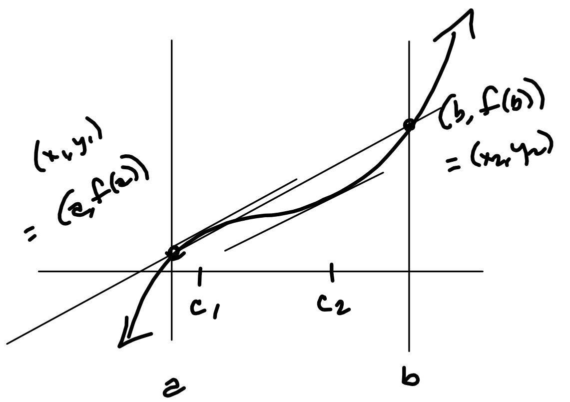
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$



2 spots shown.
MVT guarantees at least one spot

$$f'(c_1) = \frac{f(b) - f(a)}{b-a} = \frac{y_2 - y_1}{x_2 - x_1}$$

There's nothing here that says how to find the *c*. That requires *technique.*

$f(x) = 2x^2 - 4x + 5$ on $[-1, 3]$

Find $c \ni f'(c) = \frac{f(3) - f(-1)}{3 - (-1)} = \frac{11 - 11}{3 + 1} = 0$

So it's actually a Rolle's situation where $m_{avg} = 0$.

$$\begin{array}{r|rrr} 3 & 2 & -4 & 5 \\ & & 6 & 6 \\ \hline & 2 & 2 & 11 \end{array}$$

$$\begin{array}{r|rrr} -1 & 2 & -4 & 5 \\ & & -2 & 6 \\ \hline & 2 & -6 & 11 \end{array}$$

Now, solve $f'(x) = 0 \Rightarrow$

$4x - 4 = 0 \Rightarrow$ $f'(c) = 4c - 4 \overset{SET}{=} 0$, etc.

$x = \boxed{1 = c}$

$f'(1) = 4(1) - 4 = 0$

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 2x^2 - 3x + 1, \quad [0, 2]$

12. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$

13. $f(x) = \sqrt[3]{x}, \quad [0, 1]$ **14.** $f(x) = 1/x, \quad [1, 3]$

12 $f(x) = x^3 - 3x + 2$ is a polynomial. Cnts & diff'l "everywhere" $\forall x \in \mathbb{R}$

we find $c \ni$

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 0}{4} = 1$$

$$f'(c) = 3x^2 - 3 = 1 \Rightarrow$$

$$3x^2 = 4$$

$$x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{\sqrt{4}}{\sqrt{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

$\approx \pm 1.154700538$

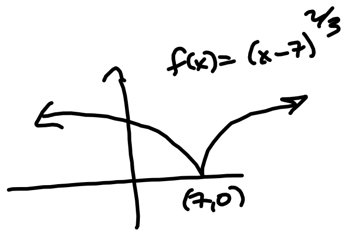
$f(x) = \frac{1}{x-1}$ on $[0,2]$

Doesn't satisfy the hypotheses, b/c it has a discontinuity @ $x = 1 \in [0,2]$.

$\therefore$ MVT Does not apply.

The hypotheses do not hold

$f(x) = (x-7)^{2/3}$ on $[5, 8]$.



f is cnts everywhere.

$$f'(x) = \frac{2}{3}(x-7)^{-\frac{1}{3}} = \frac{2}{3(x-7)^{1/3}}$$

$\nexists$ @ $x = 7$.

Proof of Rolle's Theorem

If $f(x) = f(a) = f(b)$ $\forall$ $x \in [a,b]$, we are done

Let $c =$ any $x \in (a,b)$. Then $f'(c) = 0$.

Assuming f is not the constant function $f(x) = f(a)$,

For the purposes of argument, assume $f(x) > f(a)$ somewhere in (a,b). (Otherwise, replace "maximum" with "minimum" below.)

By the Extreme Value Theorem, f achieves an absolute maximum at some $c \in [a,b]$. As $f(a) = f(b)$ is NOT the maximum value, $c \neq a$ & $c \neq b$, i.e., $c \in (a,b)$.

By Fermat's Theorem, $f'(c) = 0$, and we are done. $\blacksquare$

Proof of MVT

We build a function from $f(x)$ that satisfies Rolle's Theorem.

Define $g(x) = f(x) -$ secant line containing $(a, f(a))$ & $(b, f(b))$.

Then $g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$

Recall: $\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1) + y_1 = m(x-x_1)+y_1$

where $(x_1,y_1) = (a, f(a))$ & $(x_2,y_2) = (b, f(b))$

$f(x)$ is cnt$^{\underline{s}}$ on $[a,b]$ & the secant line is a line and therefore cnt^{s} & dif$\underline{bl}$ everywhere.

$\therefore$ $g(x)$ is cnt$^{\underline{s}}$ on $[a,b]$.

$f(x)$ is dif$\underline{bl}$ on (a,b) & so's the line, so

$g(x) = f(x) -$ line is dif$\underline{bl}$ on (a,b)

Now, $g(a) = f(a) - \left[\frac{f(b)-f(a)}{b-a}(a-a) + f(a)\right] = 0$

and $g(b) = f(b) - \left[\frac{f(b)-f(a)}{b-a}(b-a) + f(a)\right]$

$= f(b) - [f(b) - f(a) + f(a)] = f(b) - f(b) = 0 = f(a)$,

$\therefore$ $g(x)$ satisfies the hypotheses of Rolle's Theorem.

$\therefore$ $\exists\, c \in (a,b) \ni g'(c) = 0$

That's the key. Now, let's compute $g'(x)$:

$$\frac{d}{dx}\left[f(x)-\left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]\right]$$

$$=f'(x)-\left[\frac{f(b)-f(a)}{b-a}+0\right]$$

$$=f'(x)-\frac{f(b)-f(a)}{b-a}=g'(x)$$

($\frac{d}{dx}[7(x-a)]=7$. $\frac{f(b)-f(a)}{b-a}$ is just a number.)

So, $g'(c)=0 \Longrightarrow$

$$f'(c)-\frac{f(b)-f(a)}{b-a}=0$$

$\Longrightarrow f'(c)=\dfrac{f(b)-f(a)}{b-a}$ & we're done.