Reviewing for Midterm/WP#1

Compute $dy/dx = y'$

$$y = x^3 + \frac{5}{x^2} = x^3 + 5x^{-2} \longrightarrow$$

$$\boxed{y' = 3x^2 - 10x^{-3}}$$

$$y = (\overset{f}{x^2-1})(\overset{g}{\sin(x) + x^2 - 2}) \longrightarrow$$

$$\boxed{y' = \underbrace{(2x)}_{f'}\underbrace{(\sin(x) + x^2 - 2)}_{g} + \underbrace{(x^2-1)}_{f}\underbrace{(\cos(x) + 2x)}_{g'}}$$

$$y = (\overset{f}{x^2-1})^3(\overset{g}{\sin(x) + x^2 - 2})^5 \longrightarrow$$

$$\underbrace{3(x^2-1)^2(2x)}_{f'}\underbrace{(\sin(x) + x^2 - 2)^5}_{g} + \underbrace{(x^2-1)^3}_{f}\underbrace{(5(\sin(x) + x^2 - 2)^4(\cos(x) + 2x))}_{g'}$$

$$y = \frac{(x^2-1)^3}{(\sin(x) + x^2 - 2)^5} \longrightarrow$$

$$\boxed{y' = \frac{3(x^2-1)^2(2x)(\sin(x) + x^2 - 2)^5 - (x^2-1)^3(5(\sin(x) + x^2 - 2)^4(\cos(x) + 2x))}{(\sin(x) + x^2 - 2)^{10}}}$$

$$y = \sin(\cot(x^2))$$

$$y' = \cos(\cot(x^2))(-\csc^2(x^2))(2x)$$

$$\frac{d}{dx}\left[f(g(h(x)))\right] = \frac{df}{dg}\cdot\frac{dg}{dh}\cdot\frac{dh}{dx}$$

$$f(\text{☺}) = \sin(\text{☺})$$
$$g(\text{☺}) = \cot(\text{☺})$$
$$h(\text{☺}) = \text{☺}^2$$

$$x^2y^2 + x + y = \sin(x)\cos(y)$$

$$2xy^2 + 2xyy' + 1 + y' = \cos(x)\cos(y) + \sin(x)(-\sin(y))y'$$

$$x^2 \cdot 2yy' = 2x^2yy'$$

$$2x^2yy' + y' + \sin(x)\sin(y)y' = \cos(x)\cos(y) - 2xy^2 - 1$$

$$y'(2x^2y + 1 + \sin(x)\sin(y)) = \cos(x)\cos(y) - 2xy^2 - 1$$ skip?

$$\boxed{y' = \frac{\cos(x)\cos(y) - 2xy^2 - 1}{2x^2y + 1 + \sin(x)\sin(y)}}$$

Find an equation of the tangent line to $f(x) = \cos(x)$ at $x = \frac{\pi}{6}$

Sketch f and its tangent line from above.

$f(x) = \cos(x)$

$x_1 = \frac{\pi}{6}$

want $y = L_{\frac{\pi}{6}}(x) = m(x - x_1) + y_1 = f'(x_1)(x - x_1) + f(x_1)$

$= f'(\frac{\pi}{6})(x - \frac{\pi}{6}) + f(\frac{\pi}{6})$

$f'(x) = \sin(x) \to -\sin(x)$

$f'(x_1) = f'(\frac{\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2} = m \to m = -\frac{1}{2}$

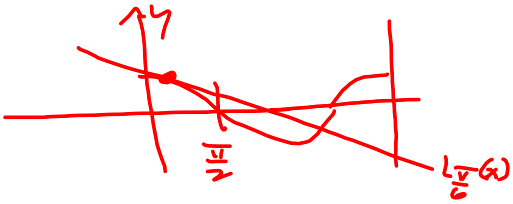
$\frac{\pi}{6} = 30^\circ$

$f(x_1) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

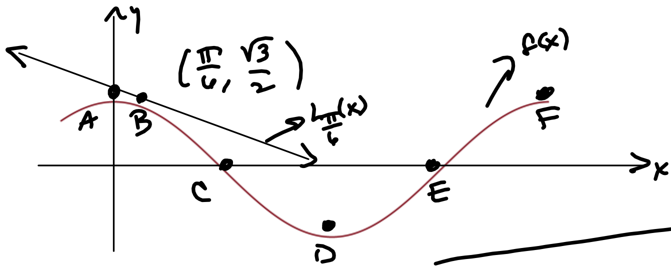
2, 1, 30°, $\sqrt{3}$

$$\boxed{L_{\frac{\pi}{6}}(x) = \frac{1}{2}(x - \frac{\pi}{6}) + \frac{\sqrt{3}}{2}}$$

$\frac{1}{2} \to -\frac{1}{2}$



$y = -\frac{1}{2}(x - \frac{\pi}{6}) + \frac{\sqrt{3}}{2}$



$A = (0, 1)$

$B = (\frac{\pi}{6}, \frac{\sqrt{3}}{2})$

$C = (\frac{\pi}{2}, 0)$

$D = (\pi, -1)$

$E = (\frac{3\pi}{2}, 0)$

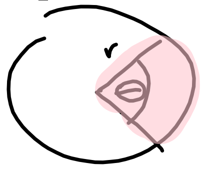
$F = (2\pi, 0)$

$$\boxed{L_{\frac{\pi}{6}}(x) = \frac{1}{2}(x - \frac{\pi}{6}) + \frac{\sqrt{3}}{2}}$$

$$\boxed{f(x) = \cos(x)}$$

35. If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

Looking at the answer, they're actuallly looking for a rate with respect to TIME.



Minute hand sweeps out area.

Rate in square cm per hour.

$A = \frac{1}{2} r^2 \theta$

$\to 0$, b/c $r = \text{constant}$

$$\frac{dA}{dt} = \boxed{2 \cdot \frac{1}{2} r \cdot \frac{dr}{dt} \cdot \theta} + \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \left(\frac{2\pi}{hr}\right)$$

$$\boxed{= \pi r^2 \frac{cm^2}{hr} = \frac{dA}{dt}}$$

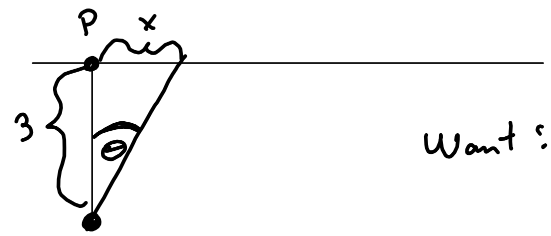
$$\boxed{\left.\begin{array}{l} \theta(0) = 0 \\ \theta(1) = 2\pi \end{array}\right\} \frac{2\pi}{hr}}$$

$\theta = \theta(t)$, where $t = $ time in hours!

Not sure you'd get this without a better-worded question.

Clearly by the book answer, this was what was intended.

44. A lighthouse is located on a small island 3 km away from the nearest point *P* on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from *P*?



$\left(\frac{4 \text{ revs}}{\text{min}}\right)\left(\frac{2\pi}{1 \text{ rev}}\right) = 8\pi \frac{\text{radians}}{\text{min}}$

Want: $\left.\frac{dx}{dt}\right|_{x=1}$

$\tan\theta = \frac{x}{3}$ $\longrightarrow$

$\frac{d}{dt}\left[\tan\theta = \frac{x}{3}\right]$ $\frac{\text{km}}{\text{km}}$

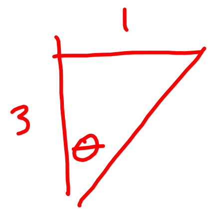
$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3}\frac{dx}{dt}$ $\longrightarrow$ $\frac{\text{km}}{\text{min}}$

$\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$

$\left.\frac{dx}{dt}\right|_{x=1} = \left.3\sec^2(\theta)\frac{d\theta}{dt}\right|_{x=1}$

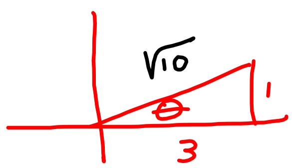
$= 3\cdot\frac{10}{9}\cdot\overset{\frac{\text{rad}}{\text{min}}}{8\pi} = \frac{80\pi}{3}\ \frac{\text{km}}{\text{min}}$

$\sec^2\left(\arctan\left(\frac{1}{3}\right)\right) = \left(\frac{\sqrt{10}}{3}\right)^2 = \frac{10}{9}$



$\tan\theta = \frac{1}{3}$ $\longrightarrow$

$\theta = \arctan\left(\frac{1}{3}\right)$



$0 < \theta < \frac{\pi}{2}$

QI

Units? Dimensional Analysis questionable

θ = angle the light beam makes with the line from the lighthouse to the point P (in radians)

t = time in minutes

x = distance from P along the shoreline. (in km)

$$\tan\theta = \frac{x}{3} \quad \frac{km}{km} \Rightarrow$$

$$3\tan\theta = x \qquad km$$

$$\frac{d}{dt}\left[\text{ above }\right]$$

$$3\sec^2\theta\,\frac{d\theta}{dt} = \frac{dx}{dt} \qquad \frac{km}{min} \checkmark$$

Use a tangent line approximation for $\sqrt{95}$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$f'(x_1) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$f(x_1) = \sqrt{100} = 10$

$L_{100}(x) = f'(x_1)(x - x_1) + f(x_1)$

$= \frac{1}{20}(x - 100) + 10$

$f'(x_1)dx = dy$!

$L_{100}(95) = \boxed{\frac{1}{20}(95 - 100)} + 10$

$= \frac{1}{20}(-5) + 10 = -\frac{1}{4} + \frac{40}{4} = \frac{39}{4} = \boxed{9.75 \approx \sqrt{95}}$

Use a differential to approximate $\sqrt{95}$

$$\left.\begin{array}{l} x_1 = 100 \\ x_2 = 95 \end{array}\right\} \Delta x = x_2 - x_1 = 95 - 100 = -5 = dx$$

$$f(x_2) \approx f'(x_1)(x - x_1) + f(x_1)$$

$$\underbrace{f(x_2) - f(x_1)}_{\text{change in } y} = \Delta y \approx dy = f'(x_1)\Delta x = f'(x_1)dx$$

$$= \frac{1}{20}(-5) = -\frac{1}{4} \approx \Delta y \text{, so}$$

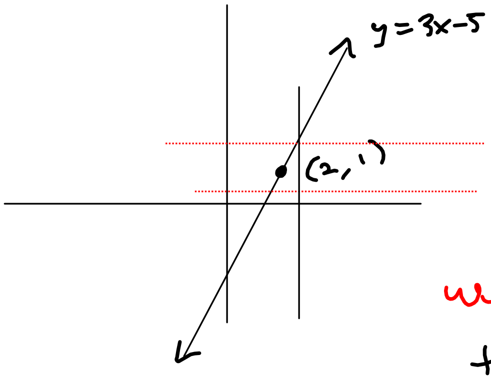
$$f(x_2) = f(x_1 + \Delta x) \approx f(x_1) \boxed{+ f'(x_1)dx} \to dy$$

$$= 10 + \left(\frac{1}{20}\right)(-5)$$

$\lim_{x\to 2}(3x-5)=1$

Challenge: Keep $3x-5$ & 1 less than ε units apart, by finding an appropriate "closeness" of x to $x=1$. → i.e. $|x-1|$ is how close x is to 1. ("How far from 1 x is.")

$|x-1| < \delta$



$\left.\begin{array}{l}1+\varepsilon\\1-\varepsilon\end{array}\right\}$ The "epsilon tube" about $y=1$.

We want to restrict $|x-1|$ so that $|3x-5-1|<\varepsilon$ *

* want $1-\varepsilon < 3x-5 < 1+\varepsilon$

$$-\varepsilon < 3x-5-1 < \varepsilon$$

$$|3x-5-1| < \varepsilon$$

$$|3x-6| < \varepsilon \longrightarrow$$

$$3|x-2| < \varepsilon \Longrightarrow$$

$$|x-2| < \frac{\varepsilon}{3} \quad \text{want.}$$

Make $\delta = \frac{\varepsilon}{3}$, so that

$$|x-2| < \delta \Longrightarrow$$

$$|3x-5-1| = |3x-6| = 3|x-2| < 3\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon$$

$\delta = \frac{\varepsilon}{m}$ Linear

Higher Degree :

$\lim_{x\to 2} x^2-3x+1 = -1$

$\delta = \min\{1, \frac{\varepsilon}{2}\}$

Scratch work :

$|x^2-3x+1-(-1)|$

$= |x^2-3x+2| = \underbrace{|x-2|}_{<\delta}\,|x-1| \cdots < \delta\cdot \overset{2}{?} < \varepsilon$

we need something that's bigger than this (but still finite).

Bound on $|x-1|$

Assume $\delta \le 1$, then

$x \to 2$ means

$1 < x < 3 \implies$

$0 < x-1 < 2 \implies$

$|x-1| < 2$!

Proof:

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{2}\}$. Then

$0 < |x-2| < \delta$ implies

$|x^2-3x+1-(-1)| = |x^2-3x+2| = |x-1||x-2|$

$< 2|x-2| < 2\delta \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon$ ∎