2.6 - Handling product rule along with implicit differentiation.

27. $x^2 - xy - y^2 = 1$, $(2, 1)$ (hyperbola)

28. $x^2 + 2xy + 4y^2 = 12$, $(2, 1)$ (ellipse)

27 $x^2 - xy - y^2 = 1$ @ $(2,1)$ (Tan. Line)

$2x - y - xy' - 2yy' = 0$

$y'(-x-2y) = -2x + y$

$y' = \frac{2x-y}{x+2y}$

$y'\Big|_{(x,y)=(2,1)} = \frac{2(2)-1}{2+2(1)} = \frac{3}{4}$

$y = \frac{3}{4}(x-2) + 1$

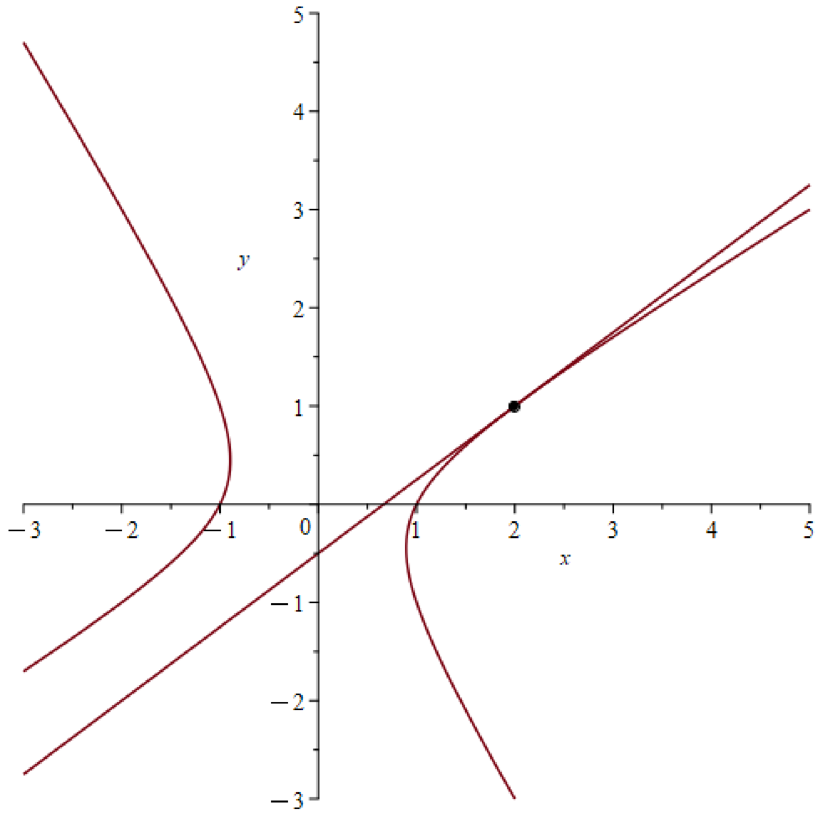
$\frac{d}{dx}[xy] = \frac{d}{dx}[x]y + x\frac{d}{dx}[y]$

$f'g + fg'$

$f = x$

$g = y$

$(2,1)$



5
4
3
2
1
0
−1
−2
−3
y
−3
−2
−1
1
2
3
4
5
x

Section 2.7

FALLING BODY'S HEIGHT ABOVE GROUND.

$h(t) = \frac{1}{2}gt^2 + v_0 t + h_0$, where

h = height in meters above ground as a function of time t, in seconds.

g = acceleration due to gravity $= -9.8 \frac{m}{s^2}$

v_0 = initial velocity

h_0 = " " height

$h'(t) = \frac{dh}{dt} = \frac{m}{s}$ = velocity at time t!

$= v(t) = gt + v_0$

$\Rightarrow h''(t) = \frac{d^2h}{dt} = g$ = acceleration! $\left(\frac{m}{s^2}\right)$

s = position

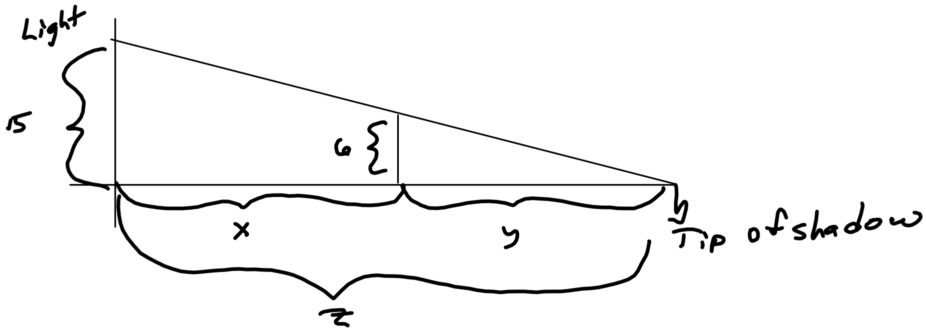
s' = velocity

s'' = acceleration

s''' = jerK

2.8 - Related Rates

15. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?



Want $\left.\frac{dz}{dt}\right|_{x=40}$ given $\frac{dx}{dt} = 5\frac{ft}{s}$

Similar triangles

$$\frac{6}{y} = \frac{15}{z} = \frac{15}{x+y} \implies 6(x+y) = 15y$$

$$6x + 6y = 15y$$

$$6x = 9y$$

$$\frac{2}{3}x = y$$

$z = x + y$

$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

Don't Know

Similar triangles

$\frac{6}{y} = \frac{15}{z} = \frac{15}{x+y}$

$6(x+y) = 15y$

$6x + 6y = 15y$

$6x = 9y$

$\frac{2}{3}x = y$

You can't hear me!

No! $\frac{dz}{dt} = \frac{5}{3} \cdot \frac{dx}{dt}$!

I forgot x was NOT the independent variable!

So $z = x + y = x + \frac{2}{3}x = \frac{5}{3}x$

$z = \frac{5}{3}x$

$\boxed{\frac{dz}{dt} = \frac{5}{3}\ \frac{ft}{s}}$

what's wrong?

$\frac{5}{3}\frac{dx}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$, Dummy!

This gets us there

$\cancel{5 + \frac{2}{3} \cdot 5}$

$\cancel{= \frac{15}{3} + \frac{10}{3} = \frac{25}{3}\ \frac{ft}{s}}$

write it up.

A man is walking away from a 15-ft light pole.

(a) $5 \frac{ft}{s}$. We find how fast the tip of the man's shadow is moving, when this 6-ft-tall man when he's 40 ft away from the pole.

Let z = the distance the tip of the shadow is from the pole (ft),

x = " " " man " " " " (ft)

y = " " " tip of the shadow is from the man (ft)

Consider the figure

We want $\left.\frac{dz}{dt}\right|_{x=40}$, given $\frac{dx}{dt} = 5\ \frac{ft}{s}$, & he's 6ft tall.

which is the rate at which the tip of his shadow is moving away from the pole when the man's 40 ft from the pole.

By similar triangles,

$$\frac{15}{x+y} = \frac{6}{y}$$

$$15y = 6x + 6y$$

$$9y = 6x$$

$$y = \frac{6}{9}x = \frac{2}{3}x \longrightarrow$$

$$z = x + y = x + \frac{2}{3}x = \frac{5}{3}x \longrightarrow \quad \frac{d}{dt}[x] = \frac{dx}{dt}$$

$$\frac{d}{dt}\left[\frac{5}{3}x\right] = \frac{5}{3}\frac{d}{dt}[x]$$

$$\frac{dz}{dt} = \frac{5}{3}\frac{dx}{dt} \longrightarrow$$

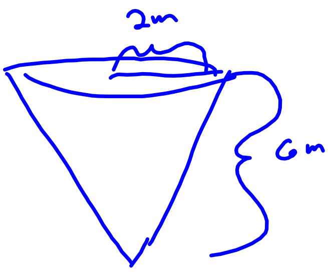
$$\left.\frac{dz}{dt}\right|_{x=40} = \frac{5}{3}\cdot 5 = \boxed{\frac{25}{3}\ \frac{ft}{s} = \left.\frac{dz}{dt}\right|_{x=40}}$$

25. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

$\cancel{\frac{dV}{dt} = -10000 \frac{\text{cm}^3}{\text{min}}}$ from leakage.

V = volume in cm^3

$\frac{dh}{dt} = \frac{20\,\text{cm}}{\text{min}}$ when $h = 2\,\text{m}$ $= 200\,\text{cm}$



Hasn't been totally written-up, yet.

$\frac{dV}{dt}$ = Rate pumped in − Rate Leaked out

$$\frac{dz}{dt} \qquad - 10000 \frac{cm^3}{min}$$

$$\frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right]$$

$$\frac{1}{3}\pi\left[\frac{d}{dt}\left[r^2 h\right]\right]$$

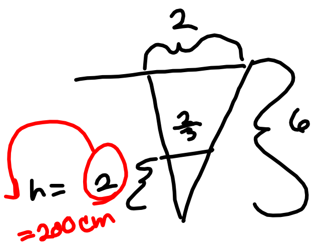
$$= \frac{1}{3}\pi\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right]$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\cdot 2\pi r\frac{dr}{dt}h + \frac{1}{3}\pi r^2\frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=2} = 20\frac{cm}{min}$$

$\Rightarrow h = 200$



$$\frac{6}{2} = \frac{2}{r} \Rightarrow 3 = \frac{2}{r} \Rightarrow$$

$$r = \frac{2}{3} = .\overline{666}\ m$$

$$= 66.\overline{6}\ cm$$

$$r = \frac{2}{3}h$$

$$\frac{dr}{dt} = \frac{2}{3}\frac{dh}{dt} = \frac{2}{3}\left(\frac{20}{\ }\right)\frac{cm}{min} = \boxed{\frac{40}{3}\ \frac{cm}{min} = \frac{dr}{dt}\Big|_{h=2}}$$

$$\frac{dV}{dt}\Big|_{h=2} = \left[\frac{1}{3}\cdot 2\pi r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt}\right]\Big|_{h=2}$$

$$= \frac{2}{3}\pi\left(\frac{2}{3}\right)\left(\frac{40}{3}\right)(2)$$

$\rightarrow 200$ $\rightarrow 200$

oops! Some of this is cm. Some is m.

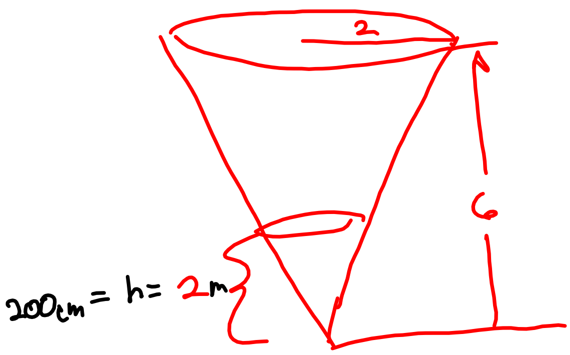
$$h = 2m = 200\,cm \longrightarrow r = \frac{2}{3}\cdot 100 = \boxed{\frac{200}{3}\,cm = r\Big|_{h=2m}}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi\left(\frac{200}{3}\right)\left(\frac{40}{3}\right)(200) + \frac{1}{3}\pi\left(\frac{200}{3}\right)^2(20)$$

$$= \cancel{\frac{1}{3}\pi\left(\frac{200}{3}\right)(20)\left[2\left(\frac{40}{3}\right)(10)\right.}$$ ~~Not helping.~~

This gives $\frac{dV}{dt}$. Subtract the leakage rate & that will give the pumping rate.

Math isn't a direct path to easy solutions. It's about trying, failing and revising your results.



height of cone = 6m = 600 cm

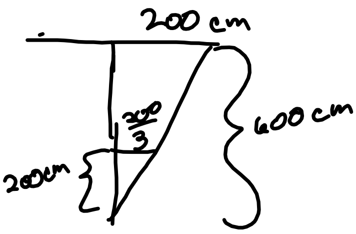
$\left.\frac{dh}{dt}\right|_{h=200\text{ cm}} = 20\ \frac{\text{cm}}{\text{min}}$

Leaking : 10,000 $\frac{\text{cm}^3}{\text{min}}$.

$\frac{dV}{dt}$ = Vol in − Vol out
= pump rate − 10000 → So, pump rate = $\frac{dV}{dt} + 10000$!
= x − 10000, where x = pump rate in $\frac{\text{cm}^3}{\text{min}}$.
= what we want.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} h + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$



$r = \frac{1}{3}h$ by similar triangles

$$\frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt} = \frac{1}{3} \cdot 20 \ \frac{cm}{min} = \boxed{\frac{20}{3} \frac{cm}{min} = \frac{dr}{dt}}$$

$$\left.\frac{dV}{dt}\right|_{h=200} = \frac{2}{3}\pi\left(\frac{200}{3}\right)\left(\frac{20}{3}\right)(200) + \frac{1}{3}\pi\left(\frac{200}{3}\right)^2(20)$$

$$= \frac{800000\pi}{9}$$, working Against the leakage.

So you must be pumping at a rate of

$$\frac{800000\pi}{9} + 10000 \ \frac{cm^3}{min}$$