Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{df}{dx}\cdot g(x) - f(x)\cdot\frac{dg}{dx}}{g(x)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Recall from last time:

Your book rearranges things. All books do. They think it helps stupid people, but it slows down smart people, in my opinion, and u r smert.

b & c are constants, f & g are differentiable functions

$$\frac{d}{dx}[c]=0,\quad \frac{d}{dx}[x^n] = nx^{n-1},\quad \frac{d}{dx}[cx]=c$$

$$\frac{d}{dx}[cf(x)+bg(x)] = c\cdot\frac{df}{dx} + b\frac{dg}{dx}$$

$$(cf+bg)' = cf'+bg'$$

"The differential operator is a linear operator." It respects addition and it respects multiplication by a constant.

Derivatives are "linear operators" because of linearity properties of limits.

$$\lim(cf+bg) = c\lim f + b\lim g$$

if b, c constant & lim f & lim g exist.

$$(fg)' = f'g+fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$$

$$\frac{d}{dx}\left[\frac{x^2-5x}{x^4+2x+1}\right] = \frac{(2x-5)(x^4+2x+1)-(x^2-5x)(4x^3+2)}{(x^4+2x+1)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$$

Example from last time.

$$f = x^2-5x \Longrightarrow f' = 2x-5$$

$$g = x^4+2x+1 \Longrightarrow 4x^3+2 = g'$$

Proof of the Quotient Rule, informally...

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{g(x)g(x+h)}\right]$$

$$= \frac{f(x+h) - f(x)}{h} \cdot \frac{g(x)}{g(x)g(x+h)} - \frac{f(x)}{g(x)g(x+h)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$\xrightarrow{h \to 0} f'(x) \cdot \frac{g(x)}{g(x)^2} - \frac{f(x)}{g(x)^2} \cdot g'(x)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Limit of the product/quotient is the product/quotient of the limits, provided the limits exist, separately, which they do, here,

by (continuity and) differentiability of *f* and *g*. We're also assuming that $g(x)$ isn't zero at the *x* in question.

Section 2.4 Derivatives of Trig Functions

Claim: $\frac{d}{dx}[\sin(x)] = \cos(x)$

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

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$$= \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

$$= \frac{\sin(x)\cos(h)-\sin(x)+\sin(h)\cos(x)}{h}$$

$$= \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \frac{\sin(h)}{h}\cos(x)$$

I'll leave this converging to zero.

I'll prove that

$\frac{\sin(h)}{h}$ need this to converge to 1

2.4 Derivative of trig functions. **Cue card for teacher.**

This will give the quotient an upper bound of 1 for small, positive θ.

$$\sin\theta < \theta$$

$\theta = h$ is a small positive angle.

This will make the quotient bigger than cosine (for small, positive θ.

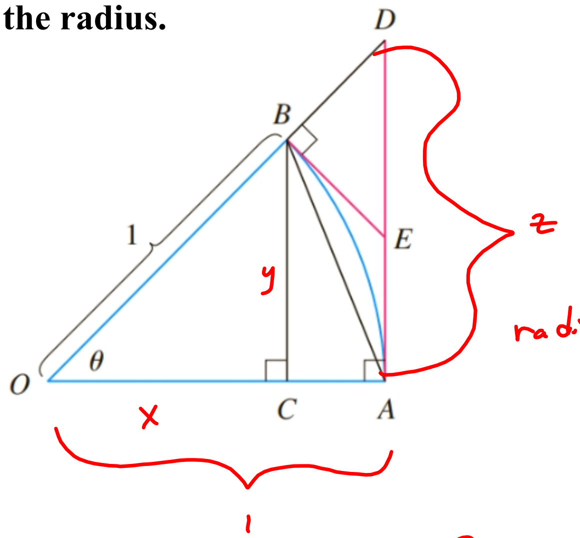
$$\theta < \tan\theta$$

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

Then we can squeeze it in the limit:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Tangent to circle is perpendicular to the radius.



We see a portion of the unit circle, the wedge corresponding to a small positive angle

This small angle corresponds to the *h* in the difference quotient for sine.

radians $s = r\theta = \theta$ when $r = 1$.

so $\sin\theta < \theta \Longrightarrow$

$$\frac{\sin\theta}{\theta} < 1$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta > \theta \Longrightarrow$$

$$\frac{\sin\theta}{\theta} > \cos\theta$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$\theta \to 0$ $\qquad$ $h \to 0$

$$1 \le \lim_{h\to 0}\frac{\sin\theta}{\theta} \le 1$$

By the Squeeze Theorem, we obtain the desired limit.

Derivatives of Trig Functions Laid Out:

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$$

$$\frac{d}{dx}\left[\cos(x)\right] = \sin(x) \qquad \text{(Similar Proof)}$$

$$\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$$

Pf :

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{f}{g}$$

$$\Rightarrow \frac{d}{dx}\left[\tan(x)\right] = \frac{f'g - fg'}{g^2} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x) = \frac{d}{dx}\left[\tan(x)\right]$$

$$\frac{d}{dx}\left[\csc(x)\right] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{f'g - fg'}{g^2} = \frac{0 \cdot \sin(x) - 1(\cos(x))}{\sin^2(x)}$$

$$f = 1,\ g = \sin(x)$$

$$= \frac{-\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} = \boxed{-\csc(x)\cot(x) = \frac{d}{dx}\left[\csc(x)\right]}$$

$$\boxed{\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\cot(x)}$$

$$\boxed{\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)}$$

$$\frac{d}{dx}\left[x^2\sin(x)\right] = 2x\sin(x) + x^2\cos(x)$$

$$\frac{d}{dx}[fg] = f'g + fg'$$

$$\frac{d}{dx}\left[\frac{x^2\sin x}{\tan(x)-1}\right] = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$f = x^2\sin(x)$, f' below

$g = \tan(x)-1$, $g' = \sec^2(x)$

$$= \frac{\left(2x\sin(x) + x^2\cos^2(x)\right)(\tan(x)-1) - \left(x^2\sin(x)\right)\left(\sec^2(x)\right)}{(\tan(x)-1)^2}$$

GOOD FINAL ANSWER

$x^2\sin(x)$ product rule:

$f = x^2$, $g = \sin(x)$

$f' = 2x$, $g' = \cos(x)$

Tangent Lines Done the Grown-Up Way

$f(x) = x^3+3x^2+x+7$

Find where *f* has horizontal tangents.

$f'(x) = 3x^2+6x+1 \overset{\text{SET}}{=} 0$

$a=3, b=6, c=1$

$b^2-4ac = 6^2-4(3)(1)$

$= 36-12 = 24$

$\sqrt{24} = 2\sqrt{6}$

$2 \,|\, 24$, $2 \,|\, 12$, $2 \,|\, 6$, 3

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-6 \pm 2\sqrt{6}}{2(3)} = \dfrac{-6 \pm 2\sqrt{6}}{6}$

$\boxed{x = \dfrac{-3 \pm \sqrt{6}}{3}} = -1 \pm \dfrac{\sqrt{6}}{3}$

WebAssign...

$3x^2+6x+1 = 0 \longrightarrow$

$3(x^2+2x \quad) = -1$

$3(\underbrace{x^2+2x+1^2}) = -1+3(1) = 2$

$\frac{2}{2} = 1 \rightarrow 1^2 = 1$

$3(x+1)^2 = 2$

$(x+1)^2 = \frac{2}{3}$

$\sqrt{(x+1)^2} = \sqrt{\frac{2}{3}}$ $\qquad \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

$|x+1| = \sqrt{\frac{2}{3}}$

$x+1 = \pm\sqrt{2/3}$

$x = -1 \pm \sqrt{\frac{2}{3}} = \boxed{-1 \pm \frac{\sqrt{6}}{3} = x}$

$$3x^2+6x+1 = 0$$

$$x^2+2x+\frac{1}{3} = 0$$

$$x^2+2x+1^2 = -\frac{1}{3}+1 = \frac{2}{3}$$

$$\frac{2}{2}=1 \rightarrow 1^2 = 1$$

$$(x+1)^2 = \frac{2}{3}$$

$$x+1 = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$$

$$x = -1 \pm \frac{\sqrt{6}}{3}$$

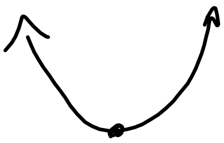
Re-write $3x^2+6x+1$ in the form $a(x-h)^2+k$

: $3x^2+6x+1 = 3(x^2+2x \quad)+1$

$= 3(x^2+2x+1^2)+1-3$

$= 3(x+1)^2-2$

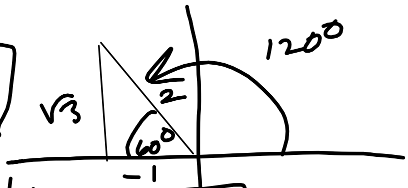
$(h,k) = (-1,2)$



(-1,2)

$f(x) = \sin(x)$. Find eq'n of tangent line to $f(x)$ @ $x = \frac{3\pi}{3}$

$f(\frac{2\pi}{3}) = \sin(\frac{2\pi}{3}) = \boxed{\frac{\sqrt{3}}{2} = f(\frac{2\pi}{3})}$



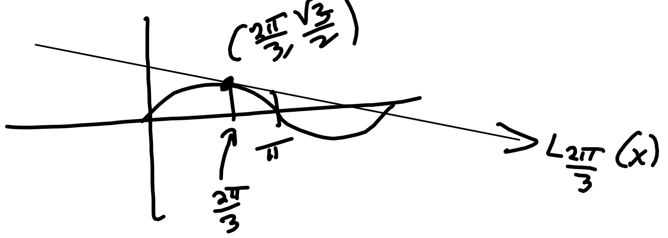
$f'(x) = \cos(x)$. $f'(\frac{2\pi}{3}) = \cos(\frac{2\pi}{3}) = \boxed{-\frac{1}{2} = f'(\frac{2\pi}{3})}$

$y = L_{\frac{2\pi}{3}}(x) =$ Linearization of f @ $\frac{2\pi}{3}$ is

$y = f'\left(\frac{2\pi}{3}\right)\left(x - \frac{2\pi}{3}\right) + f\left(\frac{2\pi}{3}\right)$

$= m(x - x_1) + y_1$

$$\boxed{y = -\frac{1}{2}\left(x - \frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2}}$$



2.5 - The Chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dg}\cdot\frac{dg}{dx}$$

$$\frac{d}{dx}\left[\sqrt{x^5-7x}\right] = \frac{d}{dx}\left[\left(x^5-7x\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left(x^5-7x\right)^{-\frac{1}{2}}\left(5x^4-7\right)$$

$$f(g) = g^{\frac{1}{2}} \qquad \frac{df}{dg} = \frac{1}{2}g^{-\frac{1}{2}} \qquad = \frac{5x^4-7}{2\sqrt{x^5-7x}}$$

$$g(x) = x^5-7x \qquad \frac{dg}{dx} = 5x^4-7$$