Section 1.8

Continuity and consequences of continuity.

f is cnt$^{\underline{s}}$ @ c if $\lim\limits_{x\to c} f(x) = f(c)$. No HOLES. NO BREAKS.

(This implies that $c \in \mathcal{D}(f)$, the way I wrote it.)

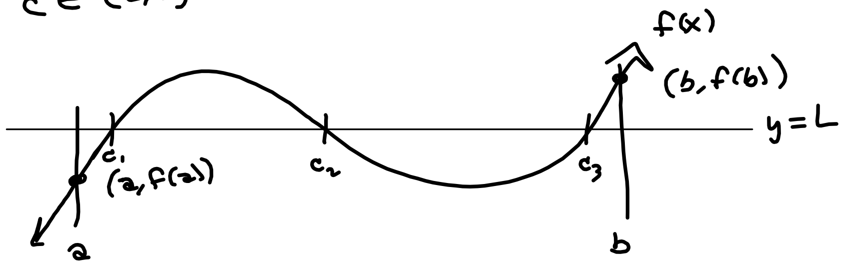
Extreme value theorem: (True, but wait 'til Chapter 3!)

f cnt$^{\underline{s}}$ on $[a,b] \Longrightarrow f$ achieves a max and a min somewhere in $[a,b]$.

Intermediate value Theorem:

f cnt$^{\underline{s}}$ on $[a,b]$ & $f(a) < L < f(b)$ (OR $f(a) > L > f(b)$)

$\Longrightarrow \exists\, c \in (a,b) \ni f(c) = L$.



Properties of cnt$^{\underline{s}}$ funcs f, g.

$f \pm g$, fg, $\frac{f}{g}$ (where $g(x) \neq 0$) are cnt$^{\underline{s}}$

$f \circ g$ is cnt$^{\underline{s}}$ @ c, if g is cnt$^{\underline{s}}$ @ c and f is cnt$^{\underline{s}}$ at $g(c)$.

Prove that:

$f(x) = \cos(x) - x^3$ has at least one real zero!

$f(0) = 1 - 0^3 = 1 > 0$

$f(10) = \cos(10) - 10^3 < 0$

$\cos(x)$ is cnt^s & x^3 is cnt^s $\forall x \in \mathbb{R} \Rightarrow$

$f(x)$ is cnt^s $\forall x$.

since $f(0) = 1 > 0$ & $f(10) < 0$, $\exists c \in (1, 10)$

$\ni f(c) = 0$.

x is cnt^s.

x^n is cnt^s $\forall n \in \mathbb{R}$

Polynomials, $\sin(x)$, $\cos(x)$, $\tan(x)$ (on its domain)

Rational functions

Pretty much every function we can write is continuous on its domaiin and continuity usually boils down to domain questions.

$\frac{\text{Stuff}}{0}$ BAD

$\sqrt[2n]{\text{negative}}$ BAD $(n \in \mathbb{N})$

13. 0/1 points

Suppose f and g are continuous functions such that $g(6) = 5$ and $\lim_{x \to 6} [3f(x) + f(x)g(x)] = 48$. Find $f(6)$.

6

$$\lim_{x \to 6} (3f(x) + f(x)g(x)) = \lim_{x\to 6}(3) \lim_{x \to c} f(x) + \lim_{x\to 6} f(x) \lim_{x \to 6} g(x)$$

$$= 3 \lim_{x\to 6} f(x) + \lim_{x \to 6} f(x) \cdot 5$$

$$= 8 \lim_{x \to c} f(x) = 48$$

$$\Longrightarrow \lim_{x \to 6} f(x) = \frac{48}{8} = 6$$

12. 0/1 points

For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 3 \\ x^3 - cx & \text{if } x \geq 3 \end{cases}$$

$c =$ 7/4

$$\lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x)$$

$$9c + 2(3) = 3^3 - 3c = \lim_{x \to 3^+} f(x)$$

$$12c = 21$$

$$c = \frac{21}{12} = \boxed{\frac{7}{4} = c}$$

#10 Why's $Q(x)$ cnt's on its domain?

$$Q(x) = \frac{\sqrt[3]{x-4}}{x^3-4}$$

Roots, quotients, sums & differences and compositions are cnt's wherever they exist.

$\mathcal{D}(Q) = \mathbb{R} \setminus \{\sqrt[3]{4}\}$ Mills Answer.

$= (-\infty, \sqrt[3]{4}) \cup (\sqrt[3]{4}, \infty)$ WebAssign Answer

15. 0/6 points

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$f(x) = x^4 + x - 5 = 0$, $(1, 2)$ cnt^s on $[1,2]$ (actually on $(-\infty, \infty)$)

$f(1) = 1^4 + 1 - 5 = -3 < 0$ $\Rightarrow$ it crosses $y=0$ line, i.e.,

$f(2) = 2^4 + 2 - 5 = 13 > 0$ implies.

$\exists c \in (1,2) \ni f(c) = 0$

there is, in, such that

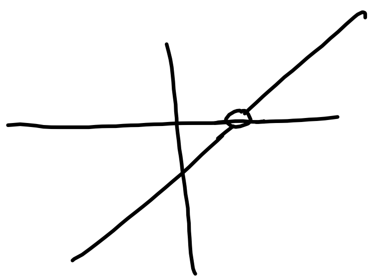
$\frac{(x-5)^2}{x-5}$ has no root in $(4,6)$

$= x-5$ $(x \neq 5)$

$\boxed{\text{Not cnt}^s \text{ @ } x=5}$ $\Rightarrow$

No zero @ $x=5$

So we can't use I.V.T. on $[4,6]$.



Prove $f(x) = \sin(x^3)$ has at least 2 x-intercepts in $(1,2)$

$x \in (1,2) \Rightarrow x^3 \in (1,8)$

$\sin(x) = 0 \quad \forall\, x = n\pi,\ n \in \mathbb{Z}$.

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π
         3.141592654
Ans*2
         6.283185307
Ans*3
         18.84955592
```

Idiot


```
         6.283185307
Ans*3
         18.84955592
Ans/6
         3.141592654
Ans*3
         9.424777961
```

2 places where $x^3 = n\pi$ for $x \in (1,2)$

$\therefore\ \sin(x^3) = 0$ in 2 spots:

$x = \sqrt[3]{\pi},\ \sqrt[3]{2\pi}$

Proof

$x^3 = \pi$ @ $x = \sqrt[3]{\pi}$ and $x^3 = 2\pi$ @ $x = \sqrt[3]{2\pi} \Rightarrow$

$\sin(x^3) = 0$ twice in $(1,2)$ ∎

Easier proof, using Intermediate value theorem and a couple of guesses:

$\sin(1)$	$\sin(1)$
$evalf(\%)$	0.8414709848
$\sin(8)$	$\sin(8)$
$evalf(\%)$	0.9893582466
$\sin(4)$	$\sin(4)$
$evalf(\%)$	-0.7568024953

So there's a zero in both $\left(1, \sqrt[3]{4}\right)$ and $\left(\sqrt[3]{4}, 2\right)$ by Intermediate Value Theorem.

63. Prove that f is continuous at a if and only if

$$\lim_{h\to 0} f(a+h) = f(a)$$

A if and only if B means

A iff B means

$A \Longleftrightarrow B$ means

means A implies B and B implies A.

Proof Suppose $\lim_{h\to 0} f(a+h) = f(a)$

" $\Longrightarrow$ "

Define $c = a+h$. Then $h \to 0 \Longrightarrow x \to c$ Nahhh...

$\lim_{h\to 0}(a+h) = a$, by properties of limits.

So we're saying $\lim_{h\to 0} f(a+h) = f(a)$

means $\lim_{x\to a} f(x) = f(a) \Longrightarrow$ cnt^s!

Now " $\Longleftarrow$ "

If f is cnt^s @ a, then $\lim_{h\to 0} f(a+h) = f(\lim_{h\to 0}(a+h))$

$= f(\lim_{h\to 0} a + \lim_{h\to 0} h) = f(a+0) = f(a) \Longrightarrow \blacksquare$

THIS IS LOGICAL EQUIVALENCE. $A \Longleftrightarrow B$

If it rains, I will bring an umbrella.

Bringing an umbrella does NOT imply it's raining.

But NOT bringing an umbrella DOES imply it is not raining!

This is the CONTRAPOSITIVE

$\cap$

$(a, f(a))$ theorem does not hold.

$\lim_{h \to 0} f(a+h) = f(a)$ iff f cnt's @ $x = a$.

Use this result to show cosine is continuous.

Prove cosine is cnt's @ a.

Let $f(x) = \cos(x)$. Then $\cos(a+h)$

$= \cos(a)\cos(h) - \sin(a)\sin(h) \xrightarrow{h \to 0}$

$\cos(a) \cdot 1 - \sin(a)\sin(0) = \cos(a)$ ∎

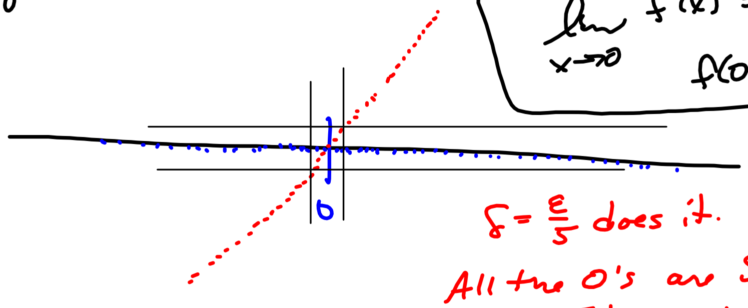
(Semi-formal. The crux of it is shown.)

#20 web Assign.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 5x & \text{if } x \text{ is irrational} \end{cases}$$

Mind-bending example to expose pathology &/or surprising consequences of the theory.

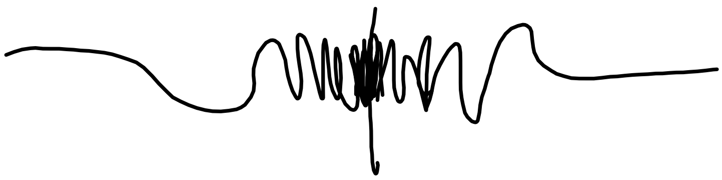
$$\boxed{\lim_{x \to 0} f(x) = 0 \ \ \& \ \ f(0) = 0!}$$



$\delta = \frac{\varepsilon}{5}$ does it.

All the 0's are in the tube.
All the 5x's are in the tube after x is within $\frac{\varepsilon}{5}$ units of 0 = x.

$f(x) = \sin\left(\frac{\pi}{x}\right)$ is <u>NOT</u> continuous @ $x=0$



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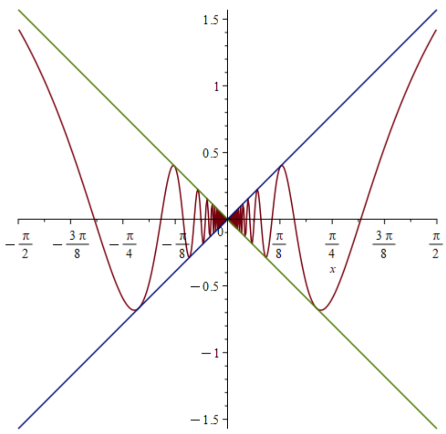
<u>BUT</u> $f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



This is either

"the topologist's sine curve"

or its evil twin.

$x^2\sin(\frac{\pi}{x})$ is closely related. It's continuous and differentiable everywhere, if we do this:

$$f(x)=\begin{cases} x^2\sin(\frac{\pi}{x}) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

Prove $x\sin(\frac{\pi}{x})$ is continuous @ $x=0$.

$$-1 \le \sin(\tfrac{\pi}{x}) \le 1 \implies$$

$$-x \le x\sin(\tfrac{\pi}{x}) \le x \quad (\text{if } x>0)$$

$$\underset{x\to 0}{\downarrow}\qquad \underset{x\to 0}{\downarrow}\qquad \underset{x\to 0}{\downarrow}$$

$$0 \qquad\quad 0 \qquad\quad 0$$

If $x<0$, then

$$-1 \le \sin(\tfrac{\pi}{x}) \le 1 \implies$$

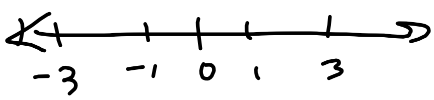
$$-x \ge x\sin(\tfrac{\pi}{x}) \ge x$$

$$\underset{x\to 0}{\downarrow}\qquad\qquad \underset{x\to 0}{\downarrow}$$

$$0 \qquad\qquad 0$$

$1<3 \implies$

$-1>-3$



#20, §1.6 Squeeze Thm.

If $f(x) < g(x)$ on (a,b)

then $\lim_{x \to b^-} f(x) \le \lim_{x \to b^-} g(x)$

$$\lim_{x \to 0} x^2 \cos\left(\frac{\pi}{x}\right) = 0$$

$$-1 \le \cos\left(\frac{\pi}{x}\right) \le 1$$

$$-x^2 \le x^2 \cos\left(\frac{\pi}{x}\right) \le x^2$$

$$\xrightarrow{x \to 0} \quad 0 \qquad 0 \qquad 0$$

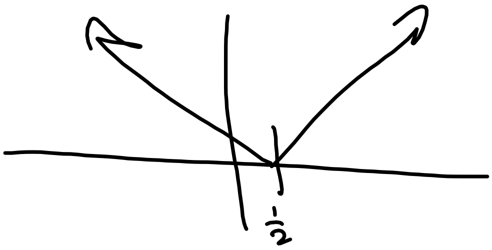
See Maple PDF from 1/25 :

https://harryzaims.com/201/201-spring-21/notes/210125-maple.pdf

$$\frac{2x-1}{|2x^3-x^2|} = \frac{2x-1}{x^2|2x-1|} = -\frac{1}{x^2} \text{ for } x<\tfrac{1}{2} \xrightarrow{x\to\frac{1}{2}^-} -\frac{1}{(\frac{1}{2})^2} = -4$$

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1<0 \end{cases}$$

$$|2x-1| = 2\left|x-\tfrac{1}{2}\right|$$



$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -(2x-1) & \text{if } x<\frac{1}{2} \end{cases}$$

Scratch:

$2x-1 \geq 0$ $\qquad$ $2x-1<0$

$2x \geq 1$ $\qquad$ $\vdots$

$x \geq \frac{1}{2}$ $\qquad$ $x<\frac{1}{2}$

$$\lim_{x\to\frac{1}{2}} \frac{2x-1}{|2x^2-x|} = \lim_{x\to\frac{1}{2}} \frac{2x-1}{|x||2x-1|} = \begin{cases} \lim_{x\to\frac{1}{2}^+} \frac{2x-1}{x(2x-1)} & \text{if } x \geq \frac{1}{2} \\ \lim_{x\to\frac{1}{2}^-} \frac{2x-1}{x(-(2x-1))} & \text{if } x<\frac{1}{2} \end{cases}$$

$$= \begin{cases} \lim_{x\to\frac{1}{2}^+} \frac{1}{x} & \text{if } x \geq \frac{1}{2} \\ \lim_{x\to\frac{1}{2}^-} -\frac{1}{x} & \text{if } x<\frac{1}{2} \end{cases} = \begin{cases} \frac{1}{\frac{1}{2}} = 2 & \text{from right} \\ \frac{1}{-\frac{1}{2}} = -2 & \text{from left} \end{cases}$$