

FTCI

$$\text{Let } g(x) = \int_0^x \frac{\sin(t)}{t^2+t+1} dt \rightarrow$$

$$g'(x) = \frac{\sin(x)}{x^2+x+1}$$

$$\text{What about } g'(x^2+2x) ? \left(\frac{dg}{d(x^2+2x)} \right) \left(\frac{d(x^2+2x)}{dx} \right)$$

$$= \frac{d}{dx} \int_0^{x^2+2x} \frac{\sin(t)}{t^2+t+1} dt = \left(\frac{\sin(x^2+2x)}{(x^2+2x)^2 + (x^2+2x) + 1} \right) (2x+2)$$

$$\int_a^b f = \int_a^0 f + \int_0^b f = - \int_0^a f + \int_0^b f$$

$$\frac{d}{dx} \int_{\sin(x)}^{\frac{x^2}{3}} \frac{t^2-5t}{\sin^2(t)} dt = \frac{d}{dx} \left[\int_{\sin(x)}^0 \frac{t^2-5t}{\sin^2(t)} dt + \int_0^{\frac{x^2}{3}} \frac{t^2-5t}{\sin^2(t)} dt \right]$$

$$= \frac{d}{dx} \left[- \int_0^{\sin(x)} \frac{t^2-5t}{\sin^2(t)} dt + \int_0^{\frac{x^2}{3}} \frac{t^2-5t}{\sin^2(t)} dt \right]$$

$$= - \left(\frac{\sin^2(x) - 5\sin(x)}{\sin^2(\sin(x))} \right) (\cos(x)) + \left(\frac{\left(\frac{x^2}{3}\right)^2 - 5\left(\frac{x^2}{3}\right)}{\sin^2\left(\frac{x^2}{3}\right)} \right) \left(\frac{2}{3}x \right)$$

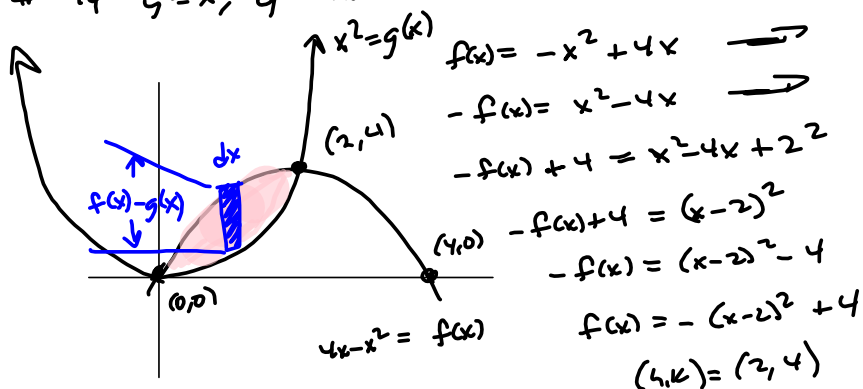
I want to have in-person final open Monday and Tuesday next week.

Part 1: Chapters 1 - 4 - 3 hrs

Part II: Chapters 5, 6 - 3 hrs?

Find the area bdd by

S.1 # 14 $y = x^2$, $y = 4x - x^2 = x(4-x)$



$$y = g(x) = x^2 \rightarrow$$

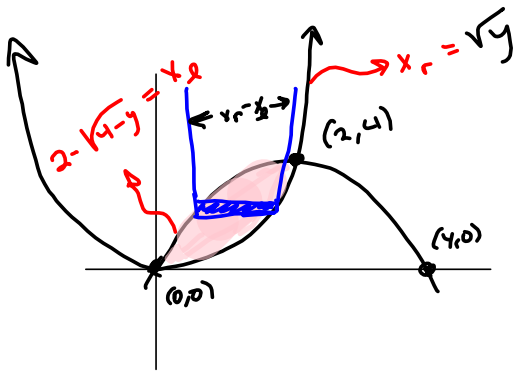
$$g(2) = 4$$

$$\text{Area} = \int_0^2 (y_u - y_l) dx$$

$$= \int_0^2 (f(x) - g(x)) dx = \int_0^2 (4x - x^2 - x^2) dx = \int_0^2 (4x - 2x^2) dx$$

$$= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2 = 2(2)^2 - \frac{2}{3}(2)^3 - (0 - 0)$$

$$= 8 - \frac{16}{3} = \frac{24-16}{3} = \boxed{\frac{8}{3} = \text{Area}}$$



$$y = 4x - x^2 = -(x-2)^2 + 4 = y$$

Solve for x :

$$-(x-2)^2 = -4 + y$$

$$(x-2)^2 = 4 - y$$

$$x-2 = \pm \sqrt{4-y}$$

$$x = 2 \pm \sqrt{4-y}$$

$$x_2 = x = 2 - \sqrt{4-y} = h(y)$$

$$y = x^2 \Rightarrow x = \pm \sqrt{y}$$

$$x = \sqrt{y} = k(y)$$

$$\text{Area} = \int_0^4 (x_r - x_l) dx$$

$$= \int_0^4 (\sqrt{y} - (2 - \sqrt{4-y})) dy = \int_0^4 (y^{\frac{1}{2}} - 2 + (4-y)^{\frac{1}{2}}) dy$$

$$= \left[\frac{2}{3} y^{\frac{3}{2}} - 2y \right]_0^4 - \int_0^4 (4-y)^{\frac{1}{2}} (-dy)$$

$$u = 4-y$$

$$du = -dy$$

$$dy = -du$$

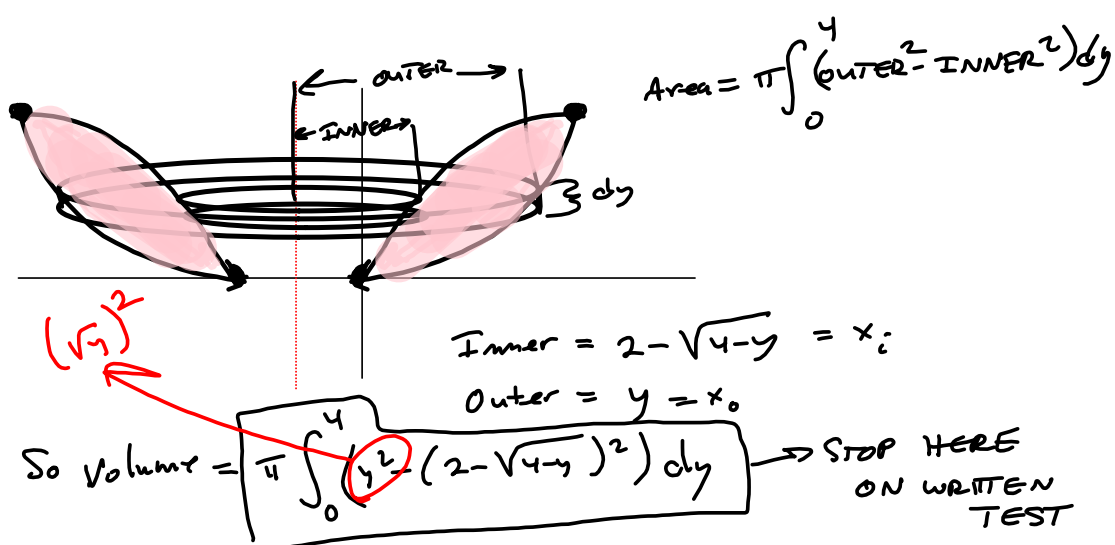
$$= \frac{2}{3}(4)^{\frac{3}{2}} - 2(4) - (0-0) - \left[\frac{2}{3}(4-y)^{\frac{3}{2}} \right]_0^4$$

$$\int (4-y)^{\frac{1}{2}} dy = \int u^{\frac{1}{2}} (-du)$$

$$= \frac{2}{3}(2)^3 - 8 - \left[\frac{2}{3}(0) - \frac{2}{3}(4)^{\frac{3}{2}} \right]$$

$$= \frac{16}{3} - \frac{24}{3} + \frac{2}{3}(2)^3 = -\frac{8}{3} + \frac{16}{3} = \boxed{\frac{8}{3} = \text{Area!}}$$

Revolve that region about $x = -3$



Bonus = $\pi \int_0^4 y^2 - (4 - 4\sqrt{4-y} + 4-y) dy$
 (come back, if time)

$$= \pi \int_0^4 (y^2 - 4 + 4\sqrt{4-y} - 4 + y) dy$$

$$= \pi \int_0^4 (y^2 + 4(4-y)^{\frac{1}{2}} + y - 8) dy$$

$$= \pi \left[\frac{1}{3}y^3 - \frac{4(4-y)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}y^2 - 8y \right]_0^4$$

$$= \pi \left[\frac{1}{3}y^3 - \frac{8}{3}(4-y)^{\frac{3}{2}} + \frac{1}{2}y^2 - 8y \right]_0^4$$

$$= \pi \left[\frac{1}{3}(4)^3 - \frac{8}{3}(0)^{\frac{3}{2}} + \frac{1}{2}(16) - 32 - \left(-\frac{8}{3}(4)^{\frac{3}{2}} \right) \right]$$

$$= \pi \left[\frac{64}{3} + 8 - 32 + \frac{8}{3}(8) \right] = \pi \left[\frac{64}{3} + \frac{64}{3} - 24 \right]$$
~~$$= \pi \left[\frac{128}{3} - \frac{72}{3} \right] = \pi \left[\frac{56}{3} \right]$$~~

$$(8 - 24 + \frac{64}{3}) \pi$$

$$= (-16 + \frac{64}{3}) \pi$$

$$= \pi \cdot \frac{-48 + 64}{3} = \frac{16}{3} \pi$$

$$= \frac{56\pi}{3} = \text{Volume}$$

consider:

$$(uv)' = u'v + uv'$$

$$\frac{d}{dx} [uv] = \left(\frac{d}{dx} u\right)v + u \frac{dv}{dx}$$

$$\int \frac{d}{dx} [uv] dx = \int \frac{du}{dx} \cdot v dx + \int u \frac{dv}{dx} dx$$

$$uv = \int v du + \int u dv \implies$$

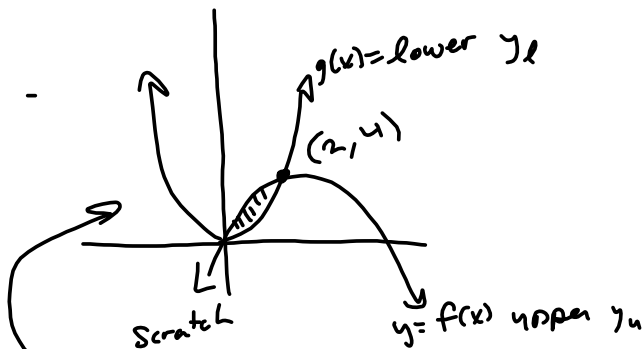
$$\boxed{\int u dv = uv - \int v du}$$

Integration by Parts
Formula.

Find the area bdd by

S.I # 14 $y = x^2$, $y = 4x - x^2 = x(4-x)$

$$\text{Area} = (\text{upper} - \text{lower}) dx = \int (g(x) - f(x)) dx$$



$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2(x^2 - 2x) = 0$$

$$x(x-2) = 0$$

$$x \in \{0, 2\}$$

$$= \int (y_u - y_l) dx$$

$$= \int_0^2 (4x - x^2 - x^2) dx$$

$$= \int_0^2 (4x - 2x^2) dx$$

$$= \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2$$

$$= \left[2x^2 - \frac{2}{3}x^3 \right]_0^2$$

$$= 2(2)^2 - \frac{2}{3}(2)^3 - [0 - 0]$$

$$= 8 - \frac{16}{3} = \frac{24-16}{3} = \boxed{\frac{8}{3} = \text{Area}}$$

Do it sideways

Scratch

$y = x^2 \rightarrow x = \pm \sqrt{y} \rightarrow x = +\sqrt{y}$ by pic.

$y = -x^2 + 4x$
 $-x^2 + 4x = y$
 $-(x^2 - 4x) = y$
 $-(x^2 - 4x + 2^2) = y - 4$
 $-(x-2)^2 = y - 4$
 $(x-2)^2 = 4 - y$
 $x - 2 = \pm \sqrt{4 - y}$
 $x = 2 \pm \sqrt{4 - y}$

$(2, 4)$ is intersection & Also the vertex of $4x - x^2$.

RIGHT-LEFT
 $= \sqrt{y} - (2 - \sqrt{4-y})$
 $= y^{\frac{1}{2}} - 2 + (4-y)^{\frac{1}{2}}$

Area = $\int_0^4 (y^{\frac{1}{2}} - 2 + (4-y)^{\frac{1}{2}}) dy$

$= \left[\frac{2}{3} y^{\frac{3}{2}} - 2y - \frac{2}{3} (4-y)^{\frac{3}{2}} \right]_0^4$

$\int (4-y)^{\frac{1}{2}} dy = -\int (4-y)^{\frac{1}{2}} dy$
 $= -\frac{2}{3} (4-y)^{\frac{3}{2}} + C$

$= \frac{2}{3} (4)^{\frac{3}{2}} - 2(4) - \frac{2}{3} (0)^{\frac{3}{2}} - \left[\frac{2}{3} (0)^{\frac{3}{2}} - 2(0) - \frac{2}{3} (4)^{\frac{3}{2}} \right]$

$= \frac{2}{3} (2)^3 - 8 + \frac{2}{3} (2)^3 = \frac{16}{3} + \frac{16}{3} - \frac{24}{3} = \frac{32-24}{3} = \frac{8}{3} = \text{Area}$

I'd like to do the area bounded by $x = 0$, $y = \sin(x)$ and $y = \cos(x)$ but we don't know our inverse trig functions yet, and the derivation requires some serious theory work:

To fully replicate the work done on WP#2, where you find the area two ways:

by integrating with respect to x and by integrating wrt y .

We could do it, but we'd have to know the antiderivatives of $\arcsin(y)$ and $\arccos(y)$.

To derive that formula requires integration by parts, which is also theory we don't yet have.

I propose a quick primer on integration by parts.

$$\int u dv = uv - \int v du$$

Consider $(uv)' = u'v + uv'$

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\rightarrow \int \frac{d}{dx}[uv] dx = \int \frac{du}{dx}v dx + \int u\frac{dv}{dx} dx$$

$$\rightarrow uv = \int v du + \int u dv$$

$$\rightarrow \boxed{\int u dv = uv - \int v du} !$$

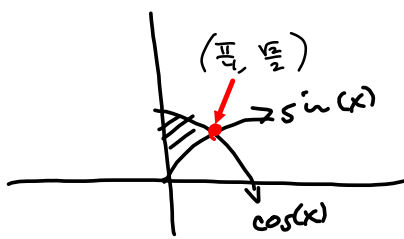
→ THE ALL-PURPOSE INTEGRATION
TOOL / LEVER

We will use this
to obtain $\int \arccos(x) dx$ & $\int \arcsin(x) dx$ &
get a head start on Calc II! For Tomorrow

$$\int \arcsin(x) dx = x \arcsin(x) + \sqrt{1-x^2} + C$$

$$\int \arccos(x) dx = x \cos(x) - \sqrt{1-x^2} + C$$

Area bdd by $f(x) = \sin(x)$, $g(x) = \cos(x)$, $x=0$



$$\text{Area} = \int_a^b (\text{upper} - \text{lower}) dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x)) dx$$

$$= \left[\sin(x) - (-\cos(x)) \right]_0^{\frac{\pi}{4}}$$

$$= \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - [\sin(0) + \cos(0)] = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - [0 + 1] = \sqrt{2} - 1 = \text{Area}$$

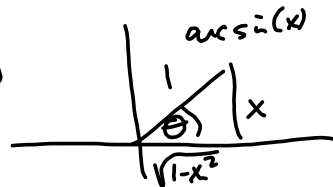
Area Between Two Curves.

To do it wrt y , we need to learn

$\sin^{-1}(x) = \arcsin(x) = \text{INVERSE SINE}$

$$\frac{d}{dx} [\arcsin(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\cos\theta}$$

$$\boxed{\begin{matrix} f(x) = \sin(x) \\ f'(x) = \cos(x) \end{matrix}}$$



$$\cos\theta = \sqrt{1-x^2} \rightarrow$$

$$\frac{d}{dx} [\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

=

$$\left. \begin{array}{l} \int \arcsin(x) dx \\ u = \arcsin(x) \\ dv = dx \end{array} \right\} \rightarrow \begin{array}{l} du = \frac{1}{\sqrt{1-x^2}} dx \\ v = x \end{array} \quad \text{scratch}$$

$$= uv - \int v du = (\arcsin(x))x - \int x \left(\frac{1}{\sqrt{1-x^2}} dx \right) = x \arcsin(x) - I.$$

$$\text{Now, } I = \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x dx)$$

$$= -\frac{1}{2} \left(\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right) + C = -\sqrt{1-x^2} + C$$

$$\text{Therefore, } \int \arcsin(x) dx = x \arcsin(x) - I$$

$$= x \arcsin(x) - (-\sqrt{1-x^2} + C) = x \arcsin(x) + \sqrt{1-x^2} + C,$$

where $C = -\hat{C}$ (to make it pretty).

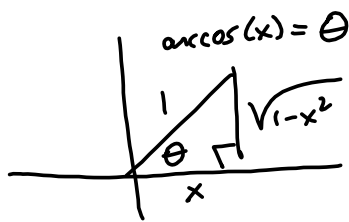
We also need $\int \arccos(x) dx$

$$\frac{d}{dx} [\arccos(x)] = \frac{d}{dx} [g^{-1}(x)] = \frac{1}{g'(g^{-1}(x))}$$

$$= \frac{1}{-\sin(\arccos(x))} = \frac{1}{-\sin \theta}$$

$$\begin{aligned} g(x) &= \cos(x) \\ g^{-1}(x) &= \arccos(x) = \cos^{-1}(x) \\ g'(x) &= -\sin(x) \end{aligned}$$

Special



$$\Rightarrow -\sin\theta = -\sqrt{1-x^2}$$

$$\Rightarrow (g^{-1})'(x) = \frac{d}{dx} [\arccos(x)] =$$

$$= \frac{1}{-\sin\theta} = \frac{1}{-\sqrt{1-x^2}} = \frac{d}{dx} [\arccos(x)]$$

Now, derive $\int \arccos(x) dx$ by
Integrating by parts.

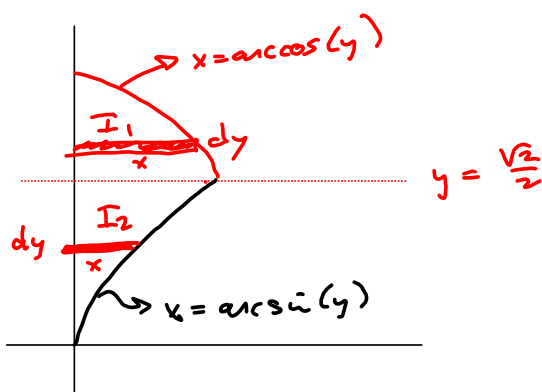
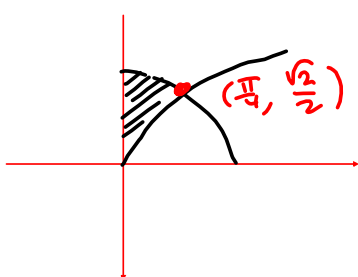
$$\left(\begin{array}{l} u = \arccos(x) \rightarrow du = -\frac{1}{\sqrt{1-x^2}} dx \\ dv = dx \rightarrow v = x \end{array} \right) \rightarrow$$

$$\int u dv = uv - \int v du = x \arccos(x) - \int x \left(-\frac{1}{\sqrt{1-x^2}} dx \right)$$

$$= x \arccos(x) - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x dx) = x \arccos(x) - \frac{1}{2} \left(\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} \right) + C$$

$$\int \arccos(x) dx = x \arccos(x) - \sqrt{1-x^2} + C$$

Now, we're ready to do this one w/ y



$$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\frac{4\sqrt{2} - \sqrt{2}\pi}{8}$$

$$\text{Area} = I_1 + I_2$$

$$\begin{aligned} I_1 &= \int_{\frac{\sqrt{2}}{2}}^1 \arccos(y) \, dy = \left[y \arccos(y) - \sqrt{1-y^2} \right]_{\frac{\sqrt{2}}{2}}^1 \\ &= \arccos(1) \sqrt{1-(1)^2} - \left[\frac{\sqrt{2}}{2} \arccos\left(\frac{\sqrt{2}}{2}\right) - \sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2} \right] \\ &= 0 - 0 - \left[\frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} - \sqrt{1-\frac{1}{2}} \right] = -\frac{\sqrt{2}\pi}{8} + \sqrt{\frac{1}{2}} = \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^{\frac{1}{\sqrt{2}}} \arcsin(y) \, dy \\ &= \left[y \arcsin(y) + \sqrt{1-y^2} \right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \arcsin\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} \\ &\quad - \left[0 + \sqrt{1} \right] = \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} + \frac{1}{\sqrt{2}} - 1 \end{aligned}$$

$$\Rightarrow I_1 + I_2 = -\frac{\sqrt{2}\pi}{8} + \frac{1}{\sqrt{2}} + \frac{\sqrt{2}\pi}{8} + \frac{1}{\sqrt{2}} - 1$$

$$\begin{aligned} &= \frac{2}{\sqrt{2}} - 1 = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2} - 1 = \text{Area the} \\ &\quad \int x \, dy \text{ way} \\ &= \int f^{-1}(y) \, dy \end{aligned}$$