Midterm Solutions, Fall, 2024

1. (5 pts each) Evaluate the following limits, if they exist. If one does not exist, explain why.

a. $\lim\limits_{x\to 2^-} \dfrac{x^2+5x-14}{|x-2|}$ b. $\lim\limits_{x\to 2^+} \dfrac{x^2+5x-14}{|x-2|}$ c. $\lim\limits_{x\to 2} \dfrac{x^2+5x-14}{|x-2|}$

Let $f(x) = \dfrac{x^2+5x-14}{|x-2|}$

(a) $\lim\limits_{x\to 2^-} \dfrac{x^2+5x-14}{|x-2|} = \lim\limits_{x\to 2^-} \dfrac{x^2+5x-14}{-(x-2)} = \lim\limits_{x\to 2^-} \dfrac{(x+7)(x-2)}{-(x-2)}$

$= \lim\limits_{x\to 2^-} \dfrac{x+7}{-1} = \dfrac{2+7}{-1} = \dfrac{9}{-1} = \boxed{-9 = \lim\limits_{x\to 2^-} f(x)}$

(b) $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \dfrac{x^2+5x-14}{|x-2|} = \lim\limits_{x\to 2^+} \dfrac{(x+7)(x-2)}{+(x-2)} = \lim\limits_{x\to 2^+} \dfrac{x+7}{1} = 2+7 = 9$

$\boxed{\lim\limits_{x\to 2^+} f(x) = +9}$

(c) By (a) & (b), we see that $\lim\limits_{x\to 2} f(x) \nexists$, because

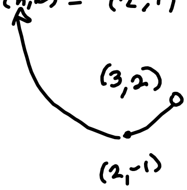
$\lim\limits_{x\to 2^-} f(x) = -9 \neq 9 = \lim\limits_{x\to 2^+} f(x)$.

2. Consider the piecewise-defined function $f(x) = \begin{cases} x^2-4x+5 & \text{if } x<3 \\ -\frac{1}{2}x+\frac{7}{2} & \text{if } x \ge 3 \end{cases}$.

a. (5 pts) Sketch the graph of $f(x)$. Label the x- and y-intercepts, the suture point(s), and the vertex of the quadratic piece, if it's in the picture. When I say "Label," I mean an ordered pair, like (0, 5), next to the point.

$x^2-4x+5 = x^2-4x+2^2-4+5 = (x-2)^2+1$

$(h,k) = (2,1)$



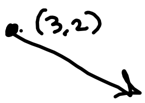
$3^2-4(3)+5 = -3+5 = 2$

$-\frac{1}{2}x+\frac{7}{2}$

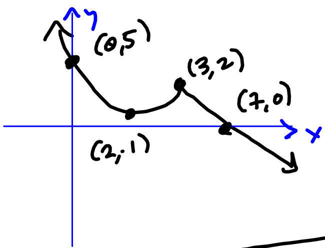
Negative slope

Plug in $x=3$:

$-\frac{1}{2}(3)+\frac{7}{2} = -\frac{3+7}{2} = \frac{4}{2} = 2$



Combine



x-ints:

$-\frac{1}{2}x+\frac{7}{2} = 0$

$-\frac{1}{2}x = -\frac{7}{2}$

$x = 7$

b. (5 pts) On what interval(s) is $f(x)$ continuous? Explain.

$\boxed{f \text{ is cont}^{\underline{s}}\ \forall x \in (-\infty,\infty)}$

Both pieces are continuous (polynomials) and they share the same suture point @ (3,2).

3. (5 pts) Simplify $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2-4x+5$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-4(x+h)+5-(x^2-4x+5)}{h}$$

$$= \frac{x^2+2xh+h^2-4x-4h+5-x^2+4x-5}{h} = \frac{2xh+h^2-4h}{h}$$

$$= \frac{h(2x+h-4)}{h} = \underset{(h\neq 0)}{2x+h-4} \xrightarrow{h\to 0} \boxed{2x-4 = f'(x)}$$

4. The point $P(2,-4)$ lies on the graph of $f(x)=x^2-2x-4$.

a. (5 pts) Write the equation of the tangent line to $f(x)$ at P.

$f'(x)=2x-2$

$f'(2)=2(2)-2=2=f'(x_1)$

$(x_1, f(x_1))=(2,-4) \longrightarrow$

$L(x)=f'(x_1)(x-x_1)+f(x_1)$

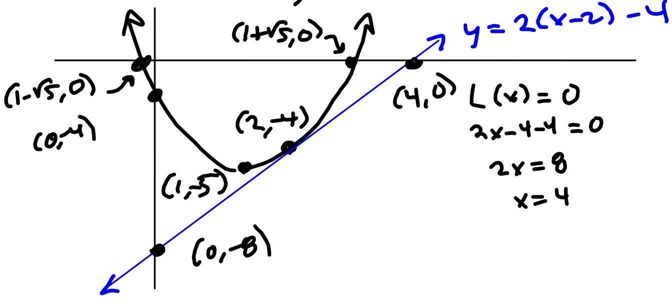
$=f'(2)(x-2)+f(2)$

$\boxed{L(x) = 2(x-2)-4}$

b. (5 pts) Sketch a graph of $f(x)$ and the tangent line to $f(x)$ at the point P.

$x^2-2x-4 = x^2-2x+1^2-1-4 = (x-1)^2-5$

$(h,k) = (1,-5)$



$L(x)=0$

$2x-4-4=0$

$2x=8$

$x=4$

$(x-1)^2=5$

$x-1=\pm\sqrt{5}$

$x=1\pm\sqrt{5}$

$f(0)=1-5=-4$

$L(0)=2(-2)-4=-8$

5. (5 pts) Sketch a plausible graph of a mostly-smooth function f that has the following properties. (Note: That very last condition is a later topic, "limits *at* infinity." So, 5 bonus points for the horizontal asymptote.)

c. $\lim_{x\to -3^-} f(x) = 7$

d. $\lim_{x\to -3^+} f(x) = -4$

e. $f(-3) = 7$

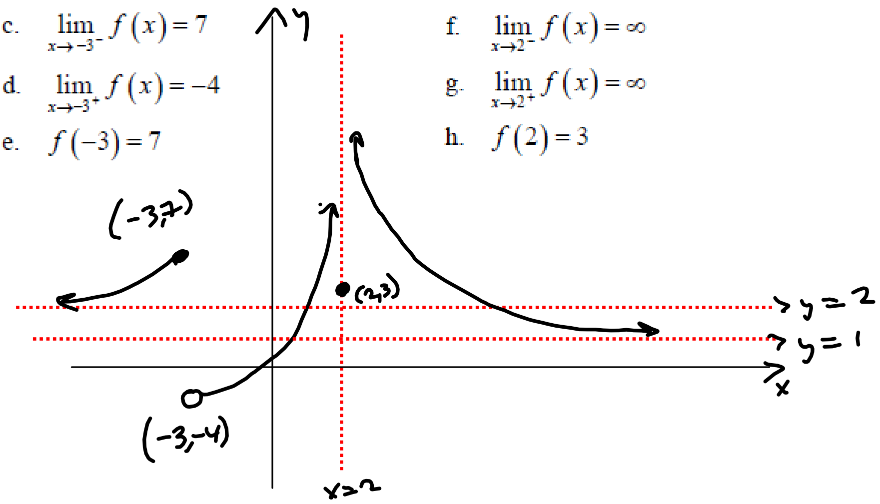
f. $\lim_{x\to 2^-} f(x) = \infty$

g. $\lim_{x\to 2^+} f(x) = \infty$

h. $f(2) = 3$

i. $\lim_{x\to \infty} f(x) = 2$

j. $\lim_{x\to \infty} f(x) = 1$



6. (5 pts) Prove that $\lim_{x\to 2}(3x-7) = -1$, using the $\varepsilon - \delta$ definition of limit.(5 pts)

Proof

Define $f(x) = 3x-7$ and $L=-1$. Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{3}$.

Then $0 < |x-2| < \delta \Rightarrow |f(x) - L| = |3x-7-(-1)| = |3x-6|$

$= 3|x-2| < 3\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon$ ■

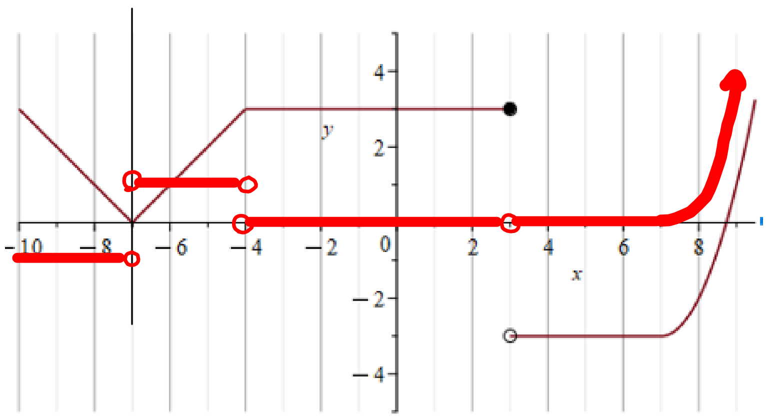
7. (5 pts) Prove that the equation $f(x) = x^2 - 4x\sin(x) + 2$ has a root in the interval $(0,2)$, but *do not solve*!

f is the sum of a polynomial and the product of x & sin(x), which are both continuous, so f is continuous everywhere.

$f(0) = 2 > 0$, $f(2) = 4 - 4(2)\sin(2) + 2 = 6 - 8\sin(2) \approx -1.274379414 < 0$

$\Rightarrow \exists\, c \in (0,2) \ni f(c) = 0$, by IVT.

8. (5 pts) The graph of a function f is given below. On the same set of axes, sketch a graph of f' This is the one problem I *want* you to write on this test sheet. Do the rest of your work for all other exercises on the blank paper provided by the proctors.



9. Differentiate the following with respect to the indicated independent variable. **Do not simplify!**

   a. (5 pts) $f(x) = \sqrt[7]{x^6} - 3x^3 + 5\sqrt{x} - \dfrac{7}{x^2}$; x.

   $$= x^{\frac{6}{7}} - 3x^3 + 5x^{\frac{1}{2}} - 7x^{-2} \longrightarrow$$

   $$\boxed{f'(x) = \frac{6}{7}x^{-\frac{1}{7}} - 9x^2 + \frac{5}{2}x^{-\frac{1}{2}} + 14x^{-3}}$$

   b. (5 pts) $g(x) = \sin(5x)\tan(3x)$; x. $\longrightarrow$

   $$\boxed{g'(x) = \cos(5x)(5)\tan(3x) + \sin(5x)\sec^2(3x)(3)}$$

   c. (5 pts) $h(\rho) = \dfrac{\cos(\rho)}{\left(7\rho^2 - 5\rho^{2/3}\right)}$; ρ. $\longrightarrow$

   $$h'(\rho) = \boxed{\frac{-\sin(\rho)(7\rho^2 - 5\rho^{2/3}) - \cos(\rho)(14\rho^2 - \frac{10}{3}\rho^{-\frac{1}{3}})}{(7\rho^2 - 5\rho^{2/3})^2}}$$

   d. (5 pts) $r(w) = \left(w^2 + 11w + 5\right)^4 (2w + 6)^3$; w. $\longrightarrow$

   $$\boxed{r'(w) = 4(w^2 + 11w + 5)^3(2w + 11)(2w + 6)^3 + (w^2 + 11w + 5)^4(3(2w + 6)^2(2))}$$

   e. (5 pts) $Q(x) = \sin(6w) - 6\sin(w)$; x. $\longrightarrow$

   $$\boxed{Q'(x) = 6\cos(6w) - 6\cos(w)}$$

10. Consider the relation $x^2 - 3xy + 4y^2 = \cos(y)$.

a. (5 pts) Use implicit differentiation to find $y' = \dfrac{dy}{dx}$

$2x - 3y - 3xy' + 8yy' = -\sin(y)\,y'$

$(-3x + 8y + \sin(y))y' = -2x + 3y$

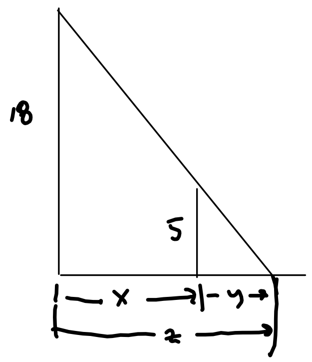
$$\boxed{y' = \frac{-2x+3y}{-3x+8y+\sin(y)}}$$

b. (5 pts) Find an equation of the tangent line to the curve at the point $(1,0)$. $= (x_1, y_1)$

$$y'\Big|_{(x,y)=(1,0)} = \frac{-2(1)+3(0)}{-3(1)+8(0)-\sin(0)} = \frac{-2}{-3} = \frac{2}{3} = m$$

$$\longrightarrow \boxed{y = \frac{2}{3}(x-1) + 0}$$

11. (5 pts) A woman who is 5 feet tall is walking away from a street light at 3 feet per second. If the light is 18 feet off the ground, how fast is the tip of the woman's shadow moving away from the light when the woman is 10 feet away? Round your final answer to 3 digits to the right of the decimal.



SIMILAR TRIANGLES

$$\frac{18}{z} = \frac{18}{x+y} = \frac{5}{y}$$

$$18y = 5x + 5y$$

$$13y = 5x$$

$$y = \frac{5}{13}x$$

$z = x + y$

want $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

. we know $\frac{dx}{dt} = 3 \text{ ft/s}$

$$z = x + y = x + \frac{5}{13}x$$

$$z = \frac{13}{13}x + \frac{5}{13}x = \frac{18}{13}x \longrightarrow$$

$$\frac{dz}{dt} = \frac{18}{13}\frac{dx}{dt}$$

$$= \left(\frac{18}{13}\right)\left(3\ \frac{\text{ft}}{\text{s}}\right) = \frac{54}{13}\ \text{ft/s}$$

$$\approx 4.153846154\ \frac{\text{ft}}{\text{s}}$$

$$\approx \boxed{4.154\ \frac{\text{ft}}{\text{s}}}$$

12. A man wants to paint the outside of a cube with a coat of paint that is 0.005 inch thick. The cube is 10 feet along one side. Use a differential to approximate the volume of paint required....

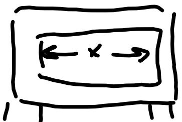
a. (5 pts) ... in cubic feet. Round your final answer to two decimal places.

Convert 0.005 in to ft : $(.005 \text{ in})\left(\frac{1 \text{ ft}}{12 \text{ in}}\right)$

Volume of cube is $V = x^3$, where x is length of one side.

The resulting length of a side after painting is

$\Delta x = 2\left(\frac{.005}{12}\right)$. If you missed that, you're off by a factor of 2.



$$\frac{.005}{12} + x + \frac{.005}{12} = x + 2\left(\frac{.005}{12}\right) = x + \frac{.01}{12}$$

$$\Delta x = \frac{.01}{12}$$

$$V = x^3 \implies dV = 3x^2 dx = 3x^2 \Delta x = 3(10^2)\left(\frac{.01}{12}\right) = \frac{3}{12} = \frac{1}{4} \text{ ft}^3$$

$$= \boxed{.25 \text{ ft}^3}$$

$\frac{.25}{2} = .125 \text{ ft}^3$

b. (5 pts) in gallons. Use 1 cubic foot = 7.48052 gallons (approximately).

$(.25)(7.48052) = 1.8701300$

$\approx \boxed{1.87 \text{ gal.}}$

$\frac{1.8701300}{2} = .935065$

$\approx .94$ gal if you missed the 2 times thing.

Bonus

1. (5 pts) Prove that $\lim_{x\to 2}\left(2x^2-3x+1\right)=3$, using the $\varepsilon-\delta$ definition of limit.

$f(x)=2x^2-3x+1,\ L=3$

$|f(x)-L|=|2x^2-3x+1-3|=|2x^2-3x-2| = |x-2||2x+1|$

$$\begin{array}{r|rrr} 2 & 2 & -3 & -2 \\ & & 4 & 2 \\ \hline & 2 & 1 & 0 \end{array}$$

Assume $\delta \le 1 \Rightarrow 1 < x < 3$

$\Rightarrow 2 < 2x < 6$

$\Rightarrow 3 < 2x+1 < 7$

$\Rightarrow |2x+1| < 7$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{7}\}$. Then if $0 < |x-2| < \delta$, we have $|f(x)-L| = |2x^2-3x+1-3|$

$= |2x^2-3x-2| = |2x+1||x-2| < 7|x-2| < 7\delta \le 7\cdot\frac{\varepsilon}{7} = \varepsilon$ ∎

2. (5 pts) Evaluate $\lim_{h \to 0} \frac{\sqrt{2x+2h} - \sqrt{2x}}{h}$, if it exists. If it does not, state why.

This is the derivative of $f(x) = \sqrt{2x} = (2x)^{\frac{1}{2}}$

$\Rightarrow$ the limit is $f'(x) = \frac{1}{2}(2x)^{-\frac{1}{2}}(2)$, by chain rule.

$\Rightarrow$ the limit is $\boxed{\frac{1}{\sqrt{2x}} = f'(x)}$

The Definition way:

$$\left(\frac{\sqrt{2x+2h} - \sqrt{2x}}{h}\right)\left(\frac{\sqrt{2x+2h} + \sqrt{2x}}{\sqrt{2x+2h} + \sqrt{2x}}\right) = \frac{2x+2h-2x}{h\left(\sqrt{2x+2h} + \sqrt{2x}\right)} = \frac{2h}{h\left(\sqrt{2x+2h} + \sqrt{2x}\right)}$$

$$= \frac{2}{\sqrt{2x+2h} + \sqrt{2x}} \xrightarrow{h \to 0} \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{2}{2\sqrt{2x}} = \boxed{\frac{1}{\sqrt{2x}}}$$

$(h \neq 0)$

3. (5 pts) See if you can *squeeze* out a *convincing* argument to support the statement

$$f(x)=\begin{cases} x^2\sin\left(\dfrac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \text{ is continuous on } (-\infty,\infty).$$

x^2 is cnt^s everywhere.

$\sin\left(\frac{\pi}{x}\right)$ is composition of a function that's cnt^s everywhere, with a function that's cnt^s everywhere, except $x=0$.

So $f(x)$ is cnt^s on $(-\infty,0)\cup(0,\infty)$.

Continuity at $x=0$ is trickier. For this, it will suffice to show that $\lim_{x\to 0} f(x) = f(0) = 0$.

To this end, observe that $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \quad \forall x \neq 0$.

since $x^2 > 0 \;\forall x \neq 0$,

$$\underset{\substack{\downarrow \\ 0}}{-x^2} \le \underset{\substack{\downarrow \\ 0}}{x^2\sin\left(\frac{\pi}{x}\right)} \le \underset{\substack{\downarrow \\ 0}}{x^2} \quad \forall x \neq 0$$

$x \to 0$ $\qquad x \to 0$ $\qquad x \to 0$

By the Squeeze Theorem,

$\lim_{x\to 0} f(x) = 0$. This means the limit agrees with the function value @ $x=0$, which is the definition of continuity at $x=0$. $\therefore$ $f(x)$ is cnt^s @ $x=0$ & so $f(x)$ is cnt^s $\forall x \in (-\infty,\infty)$ ∎