Midterm Solutions, Fall, 2024

① Let $f(x)=\frac{x^2-x-12}{|x-4|}$

ⓐ $\lim\limits_{x\to 4^+}\frac{x^2-x-12}{|x-4|} = \lim\limits_{x\to 4^+}\frac{(x-4)(x+3)}{x-4} = \lim\limits_{x\to 4^+}(x+3) = \boxed{7 = \lim\limits_{x\to 4^+} f(x)}$

ⓑ $\lim\limits_{x\to 4^-}\frac{x^2-x-12}{|x-4|} = \lim\limits_{x\to 4^-}\frac{(x-4)(x+3)}{-(x-4)} = \lim\limits_{x\to 4^-}\frac{(x+3)}{-1} = \frac{7}{-1} = \boxed{-7 = \lim\limits_{x\to 4^-} f(x)}$

ⓒ $\lim\limits_{x\to 4}\frac{x^2-x-12}{|x-4|} = \lim\limits_{x\to 4} f(x)\ \boxed{\nexists,\ b/c\ \lim\limits_{x\to 4^+} f(x) = 7 \neq -7 = \lim\limits_{x\to 4^-} f(x)}$

② $f(x)=\begin{cases} x^2+2x-15 & \text{if } x<2 \\ 6x-19 & \text{if } x\geq 2\end{cases}$

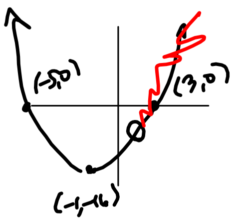
a. (5 pts) Sketch the graph of $f(x)$. Label the x- and y-intercepts, the suture point(s), and the vertex of the quadratic piece, if it's in the picture. When I say "Label," I mean an ordered pair, like (0, 5), next to the point.

$f(x)= x^2+2x-15 = (x+5)(x-3) = 0 \Rightarrow x\in\{-5, 3\}$

$f'(x) = 2x+2 = 0 \Rightarrow 2x=-2 \Rightarrow x=-1$

$f(-1) = 1-2-15 = -16 \rightsquigarrow (-1,-16)$ MIN

Check: $x^2+2x+1^2-1-5 = (x+1)^2-16 \quad (-1,-16)=(h,k)$ ✓



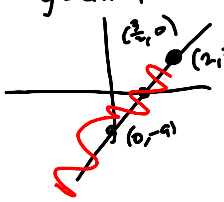
Suture Point: $x=2$

$x<2 \Rightarrow$ open dot

$2^2+2(2)-15=-7$

$(2,-7)$

$y=6x-9$



$6x=9$

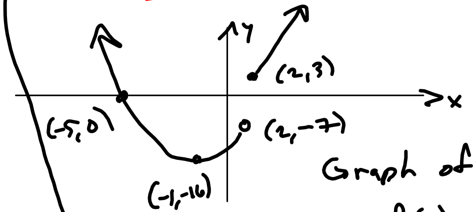
$x=\frac{9}{6}=\frac{3}{2}$

Suture $x=2$

closed dot

$6(2)-9=3$

COMBINE:



Graph of $f(x)$

b. (5 pts) On what interval(s) is $f(x)$ continuous? Explain.

f is cnts on $(-\infty,2)\cup(2,\infty)$

There's a jump discontinuity @ $x=2$:

$\lim\limits_{x\to 2^-} f(x) = -7 \neq 3 = \lim\limits_{x\to 2^+} f(x)$

3. (5 pts) Simplify $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=$ x^2-3x-4

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-3(x+h)-4-(x^2-3x-4)}{h}$$

$$= \frac{x^2+2xh+h^2-3x-3h-4-x^2+3x+4}{h} = \frac{2xh+h^2-3h}{h} = 2x+h-3 \xrightarrow{h\to 0} \boxed{2x-3}$$

$(h \neq 0)$

4. The point $P(2,-4)$ lies on the graph of $f(x)=x^2-2x-4$.

a. (5 pts) Write the equation of the tangent line to $f(x)$ at P.

$f'(x)=2x-2$

$f'(2)=2(2)-2=2=f'(x_1)$

$(x_1, f(x_1))=(2,-4) \longrightarrow$

$L(x)=f'(x_1)(x-x_1)+f(x_1)$

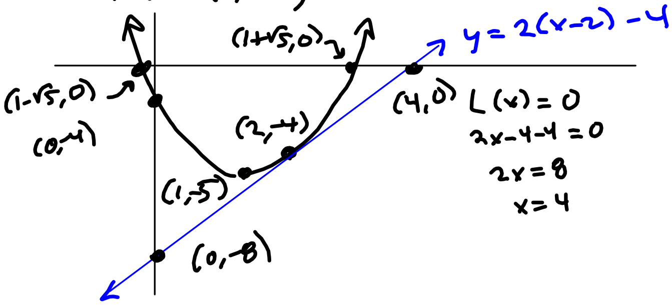
$=f'(2)(x-2)+f(2)$

$\boxed{L(x) = 2(x-2)-4}$

b. (5 pts) Sketch a graph of $f(x)$ and the tangent line to $f(x)$ at the point P.

$x^2-2x-4 = x^2-2x+1^2-1-4 = (x-1)^2-5$

$(h,k)=(1,-5)$



$L(x)=0$

$2x-4-4=0$

$2x=8$

$x=4$

$(x-1)^2=5$

$x-1=\pm\sqrt{5}$

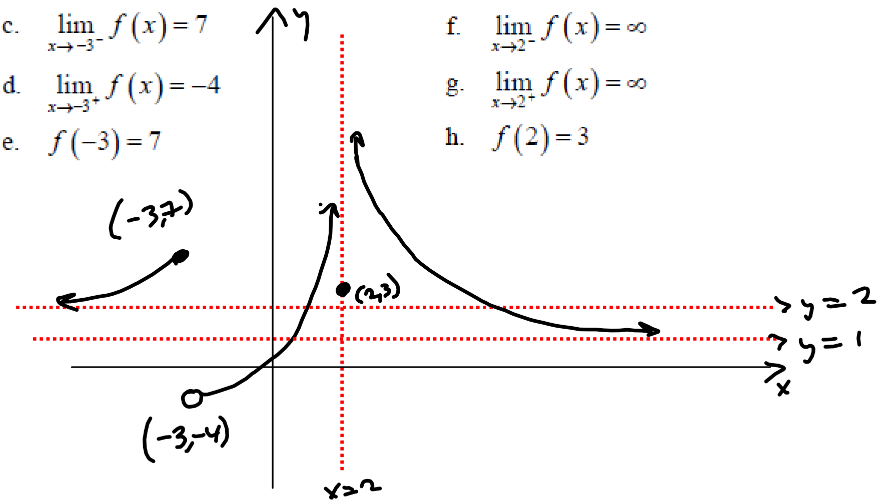
$x=1\pm\sqrt{5}$

$f(0)=1-5=-4$

$L(0)=2(-2)-4=-8$

5. (5 pts) Sketch a plausible graph of a mostly-smooth function f that has the following properties. (Note: That very last condition is a later topic, "limits *at* infinity." So, 5 bonus points for the horizontal asymptote.)

c. $\lim_{x\to -3^-} f(x) = 7$
d. $\lim_{x\to -3^+} f(x) = -4$
e. $f(-3) = 7$
f. $\lim_{x\to 2^-} f(x) = \infty$
g. $\lim_{x\to 2^+} f(x) = \infty$
h. $f(2) = 3$
i. $\lim_{x\to\infty} f(x) = 2$
j. $\lim_{x\to\infty} f(x) = 1$



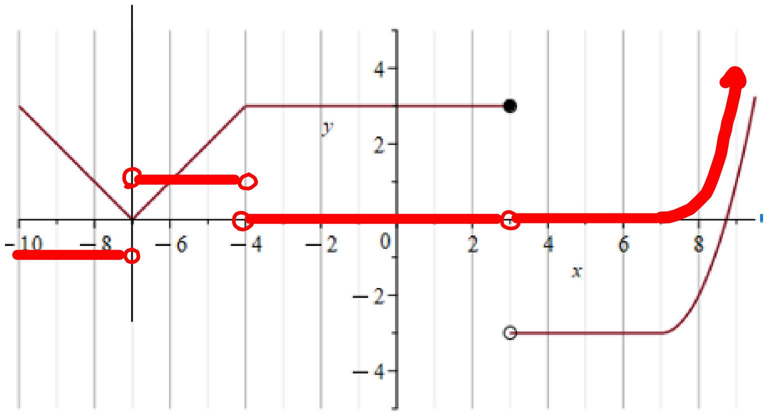
6. (5 pts) Prove that $\lim_{x\to 2}(3x-7) = -1$, using the $\varepsilon-\delta$ definition of limit.(5 pts)

Proof
Define $f(x) = 3x-7$ and $L=-1$. Let $\varepsilon>0$ be given. Define $\delta = \frac{\varepsilon}{3}$.
Then $0<|x-2|<\delta \Rightarrow |f(x)-L| = |3x-7-(-1)| = |3x-6|$
$= 3|x-2| < 3\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon$ ∎

7. (5 pts) Prove that the equation $f(x) = x^2 - 4x\sin(x) + 2$ has a root in the interval $(0,2)$, but *do not solve*!

f is the sum of a polynomial and the product of x & $\sin(x)$, which are both continuous, so f is continuous everywhere.
$f(0) = 2 > 0$, $f(2) = 4 - 4(2)\sin(2) + 2 = 6 - 8\sin(2) \approx -1.274379414 < 0$
$\Rightarrow \exists\, c \in (0,2) \ni f(c) = 0$, by IVT.

8. (5 pts) The graph of a function f is given below. On the same set of axes, sketch a graph of f' This is the one problem I *want* you to write on this test sheet. Do the rest of your work for all other exercises on the blank paper provided by the proctors.



9. Differentiate the following with respect to the indicated independent variable. **Do not simplify!**

a) $f(x) = x^{\frac{11}{4}} - 3x^4 + 5x^{\frac{1}{2}} - 11x^{-3}$

$$\boxed{f'(x) = \frac{11}{4}x^{\frac{7}{4}} - 12x^3 + \frac{5}{2}x^{-\frac{1}{2}} + 33x^{-4}}$$

b. (5 pts) $g(x) = \sin(5x)\tan(3x)$; x.

$$\boxed{g'(x) = \cos(5x)(5)\tan(3x) + \sin(5x)\sec^2(3x)(3)}$$

c) $h(p) = \dfrac{\sin(p)}{5p^3 - 4p^{3/2}}$

$$h'(p) = \boxed{\frac{\cos(p)(5p^3 - 4p^{3/2}) - \sin(p)(15p^2 - 6p^{\frac{1}{2}})}{(5p^3 - 4p^{\frac{3}{2}})^2}}$$

d) $r(w) = (w^3 + 2w + 5)^6(3w + 6)^5$

$$\boxed{r'(w) = 6(w^3+2w+5)^5(3w^2+2)(3w+6)^5 + (w^3+2w+5)^6(5(3w+6)^4(3))}$$

10 a $3x^2 - x^3y^2 = 2xy$

$6x - 3x^2y^2 - 2x^3yy' = 2y + 2xy'$

$-2x^3yy' - 2xy' = 2y - 6x + 3x^2y^2$

$$y' = \frac{2y - 6x + 3x^2y^2}{2x^3 - 2x}$$

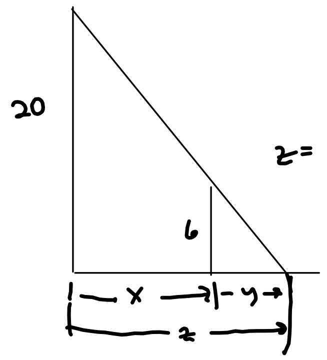
b. (5 pts) Find an equation of the tangent line to the curve at the point (~~1,0~~). (1, 1)

$(x,y) = (1,1)$

$y' = \frac{2-6+3}{2-2}$ is undefined!

Tangent line is vertical! $\boxed{x=1}$ is its eq'n!

A man who is 6 feet tall is walking away from a street light at 4 feet per second. If the light is 20 feet off the ground, how fast is the tip of the man's shadow moving away from the light when the man is 10 feet away from the light pole? Round your final answer to 3 digits to the right of the decimal.



$x =$ distance from man to light (in ft)

$y =$ " " " " tip of shadow (ft)

$z = x + y =$ " " light " " " " (in ft)

Given $\frac{dx}{dt} = 4 \frac{ft}{s}$.

Want $\left.\frac{dz}{dt}\right|_{x=10}$

$z = x + y$

Want $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

SIMILAR TRIANGLES

$\frac{20}{x+y} = \frac{6}{y} \longrightarrow$

$20y = 6x + 6y \longrightarrow$

$14y = 6x$

$\Rightarrow y = \frac{6}{14}x = \frac{3}{7}x$

$\Rightarrow z = x + y = x + \frac{3}{7}x = \frac{10}{7}x$

$\longrightarrow \frac{dz}{dt} = \frac{10}{7}\frac{dx}{dt} = \frac{10}{7}(4) = \boxed{\frac{40}{7} \frac{ft}{s}}$ regardless of distance from light pole!

1. The state of New Hampshire wants to paint the outside of a hemispherical dome of its capitol building with a coat of paint that is 0.005 inch thick. The radius of the dome is 25 feet. Use a differential to estimate the amount of paint it will take. Use the fact that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

(7 pts) Compute the volume in cubic feet. Round your final answer to two decimal places.

Volume of hemisphere is $\frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = V(r) \longrightarrow$

$$dV = 3\left(\frac{2}{3}\right)\pi r^2\,dr = 2\pi r^2\,dr$$

$$dr = \Delta r = (.005 \text{ inches})\left(\frac{1 \text{ ft}}{12 \text{ in}}\right)$$

$r = 25$ ft $\longrightarrow$

$$\Delta V = \text{Volume of paint} \approx dV = 2\pi(25)^2\left(\frac{.005}{12}\right) \text{ ft}^3$$

$$= 2\pi(625)\left(\frac{5}{12000}\right) = 625\left(\frac{5}{6000}\right)\pi =$$

$$\frac{(125)(5)}{1200}\pi = \frac{25(5)}{240}\pi = \frac{25}{48}\pi \approx \boxed{1.636246174 \text{ ft}^3}$$

b. (5 pts) in gallons. Use 1 cubic foot = 7.48052 gallons (approximately).

$$(1.636246174)(7.48052) \approx \boxed{12.23997223 \text{ gal.}}$$

Bonus

3. (5 pts) See if you can *squeeze* out a *convincing* argument to support the statement

$$f(x)=\begin{cases} x^2\sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ is continuous on $(-\infty,\infty)$.

x^2 is cnt^s everywhere.

$\sin\left(\frac{\pi}{x}\right)$ is composition of a function that's cnt^s everywhere, with a function that's cnt^s everywhere, except $x=0$.

So $f(x)$ is cnt^s on $(-\infty,0)\cup(0,\infty)$.

Continuity at $x=0$ is trickier. For this, it will suffice to show that $\lim_{x\to 0} f(x) = f(0) = 0$.

To this end, observe that $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \quad \forall x \neq 0$.

since $x^2 > 0 \ \forall x \neq 0$,

$$\underbrace{-x^2}_{\substack{x\to 0 \\ \downarrow \\ 0}} \le x^2 \sin\left(\frac{\pi}{x}\right) \le \underbrace{x^2}_{\substack{x\to 0 \\ \downarrow \\ 0}} \quad \forall x \neq 0$$

By the Squeeze Theorem,

$\lim_{x\to 0} f(x) = 0$. This means the limit agrees with the function value @ $x=0$, which is the definition of continuity at $x=0$. $\therefore$ $f(x)$ is cnt^s @ $x=0$ & so $f(x)$ is cnt^s $\forall x \in (-\infty,\infty)$ ∎

BONUS

B1 $\lim_{x\to -5}(x^2-4x+1) = 46 = \lim_{x\to -5} f(x)$ Check: $f(-5) = 25+20+1 = 46$ ✓

Scratch:

$|x^2-4x+1-46| = |x^2-4x-45| = |(x-9)(x+5)|$

$(x-9)$: Need a bound. $(x+5)$: $< \delta$

Assume $\delta \le 1$. Then $0 < |x-5| < \delta \Rightarrow$

$|x-5| < 1 \Rightarrow$

$-1 < x-5 < 1 \Rightarrow$

$-6 < x-9 < -3 \Rightarrow$

$|x-9| < 6$

<u>Proof</u>:

Let $\varepsilon > 0$ be given. Define $\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}$. Then

$0 < |x-(-5)| < \delta$ implies $|f(x)-46| = |x^2-4x+1-46|$

$= |x^2-4x-45| = |x-9||x+5| < |x-9|\,\delta < 6\delta \le 6\left(\frac{\varepsilon}{6}\right) = \varepsilon$ ∎