Ray's Office: HOR 103

Testing Room: Suite 107

Open hours: 8 am - 8 pm

RocketBook for making PDFs out for cheap.

Free "Scannable" app for iPhone (Evernote Brand)

I've seen good results from Camscanner for Android.

GeniusScan app

Notability app. Save on iPad and just upload to D2L.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(fg)' = f'g + fg'$$

General Power Rule:

$$\frac{d}{dx}\left[f(x)^n\right] = n\, f(x)^{n-1} \cdot f'(x)$$

$$\frac{d}{dx}\left[(x^2-5x)^7\right] = 7(x^2-5x)(2x-5)$$

Chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{df}{dg} \cdot \frac{dg}{dx} = f'(g)\, g'(x)$$

1. (10 pts) Evaluate $\lim_{x\to 3}\frac{x^2-4}{x^3-8}$ by factoring. ? Don't need to factor, b/c it's cont's @ $x=3$. Typo in the assignment. Wanted $\lim_{x\to 3}\frac{x^2-9}{x^3-27}$.

$$\lim_{x\to 3}\frac{x^2-4}{x^3-8} = \frac{9-4}{27-8} = \boxed{\frac{5}{19} = \lim_{x\to 3}\frac{x^2-4}{x^3-8}}$$

. (10 pts) Evaluate each of the following by factoring and simplifying. One exists. The other doesn't.

a. $\lim_{x\to -7}\frac{3x^2+17x-28}{4x^2+31x+21} = \lim_{x\to -7} f(x)$

$$f(x) = \frac{(3x-4)(x+7)}{(4x+3)(x+7)} = \frac{3x-4}{4x+3} \xrightarrow{x\to -7} \frac{-21-4}{-28+3} = \frac{-25}{-25} = \boxed{1 = \lim_{x\to -7} f(x)}$$

$(x \neq -7)$

Scratch for factoring:

$(2)(2)(3)(7)$

$28+3=31$

$4x^2+31x+21$

$= 4x^2+28x+3x+21$

$= 4x(x+7)+3(x+7)$

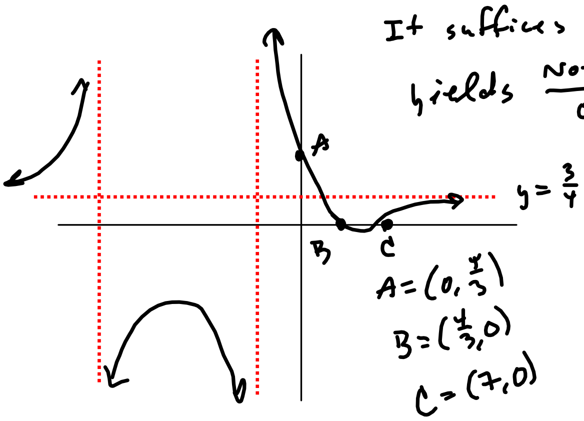
$(x+7)(4x+3)$

b. $\lim_{x\to -7}\frac{3x^2-25x+28}{4x^2+31x+21} = \lim_{x\to -7} f(x)$ DNE, b/c $x=-7$ is vertical asymptote

$$f(x) = \frac{3x^2-25x+28}{4x^2+31x+21} = \frac{(3x-4)(x-7)}{(4x+3)(x+7)}$$

$x=-7$ is a vertical asymptote for $f(x)$, which looks like this:

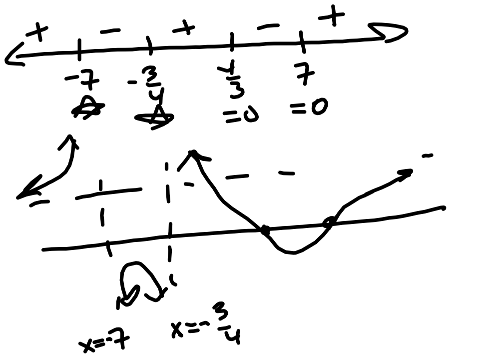
It suffices to observe that direct substitution yields $\frac{\text{Not } 0}{0}$.



$A = (0, \frac{4}{3})$

$B = (\frac{4}{3}, 0)$

$C = (7, 0)$



3. (10 pts) Prove that $\lim_{x\to 5}(7x-3)=32$. (This is the $\varepsilon-\delta$ proof you're dying to do.)

$f(x)=7x-3$, $L=32$.

Slope $=7=\frac{\Delta y}{\Delta x}$. want $\Delta y=|f(x)-L|<\varepsilon$ when $\Delta x=|x-5|<\delta$

$$7\Delta x=\Delta y$$
$$\Delta x=\frac{\Delta y}{7}$$
$$\delta=\frac{\varepsilon}{7}$$

Proof

Let $\varepsilon>0$ be given. Define $\delta=\frac{\varepsilon}{7}$. Then $0<|x-5|<\delta \Rightarrow$

$|f(x)-L|=|7x-3-32|=|7x-35|=7|x-5|<7\delta=7\left(\frac{\varepsilon}{7}\right)=\varepsilon$ ∎

4. (10 pts) Compute the derivative of $f(x)=x^2-3x+4$ by the definition of derivative. This means taking the limit of a difference quotient.

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-3(x+h)+4-(x^2-3x+4)}{h}$$

$$=\frac{x^2+2xh+h^2-3x-3h+4-x^2+3x-4}{h}$$

$$=\frac{2xh+h^2-3h}{h}=\frac{h(2x+h-3)}{h}\underset{h\neq 0}{=}2x+h-3\xrightarrow{h\to 0}\boxed{2x-3=f'(x)}$$

5. (5 pts each) Compute the derivatives of each of the following. Do not simplify your answer.

a. $y=3x^2-6x+\frac{7}{x^3}$ $= 3x^2-6x+7x^{-3} \Longrightarrow \boxed{y' = 6x-6-21x^{-3}}$

b. $y=\left(x^3+5x\right)\left(4x-1\right)$ $\Longrightarrow y' = f'g+fg' = \boxed{(3x^2+5)(4x-1)+(x^3+5x)(4) = y'}$

c. (5 pts.) $y=\dfrac{x^2+5x}{7x-1}$ $= \dfrac{f}{g} \Longrightarrow y' = \dfrac{f'g-fg'}{g^2}$

Same

$$= \boxed{\frac{(2x+5)(7x-1)-(x^2+5x)(7)}{(7x-1)^2}}$$

d. $y=\left(x^3+7x\right)^4\left(4x-1\right)^{11}$ $= f^4g^{11} \Longrightarrow y' = 4f^3f'g^{11}+f^4(11g^{10}g')$

$$= \boxed{4(x^3+7x)^3(3x^2+7)(4x-1)^{11}+(x^3+7x)^4(11(4x-1)^{10}(4))}$$

e. $y = \csc^2(6x-7) = \left(f(g(x))\right)^2$, where $f(g(x)) = \csc(6x-7)$

$\Rightarrow y' = 2f(g(x))^1\left(\frac{df}{dg}\cdot\frac{dg}{dx}\right)$

$= \left(2\csc(6x-7)\right)\left(-\csc(6x-7)\cot(6x-7)\right)(6)$

f. $y = \sin\left(\tan\left(x^2-5\right)\right) = f(g(h(x)))$, where $f(g) = \sin(g)$

$g(h) = \tan(h)$ &

$h(x) = x^2 - 5$

$\Rightarrow y' = \frac{df}{dg}\cdot\frac{dg}{dh}\cdot\frac{dh}{dx} = \boxed{\left(\cos\left(\tan(x^2-5)\right)\right)\left(\sec^2(x^2-5)\right)(2x) = y'}$

6. (10 pts) Find an equation of the tangent line to $f(x)=\cos(x)$ at $x=\frac{\pi}{4}$. Then sketch the graph of this situation, with the function and its tangent line, together on the same set of axes.

tangent line $= y = L(x) = f'(x_1)(x-x_1)+y_1$

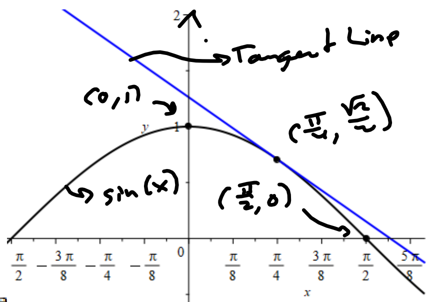
$f(x)=\cos(x)$, $x_1=\frac{\pi}{4}$ $\rightarrow$

$f(x_1)=\cos(x_1)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$, either way is fine.

$\Rightarrow f'(x_1)=-\sin(x_1)=-\sin\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$.

$\Rightarrow L(x)=f'(x_1)(x-x_1)+f(x_1)$

$$=\boxed{-\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}=L(x)}$$



7. (5 pts) Use your result from the previous problem to approximate $\cos(40°)$

$40° = (40°)\left(\frac{\pi}{180}\right) = \frac{2\pi}{9}$

$L\left(\frac{2\pi}{9}\right) = -\frac{1}{\sqrt{2}}\left(\frac{2\pi}{9}-\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(\frac{8\pi-9\pi}{36}\right)+\frac{1}{\sqrt{2}}$

$= -\frac{\sqrt{2}}{2}\left(-\frac{\pi}{36}\right)+\frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi}{72}+\frac{\sqrt{2}}{2}\cdot\frac{36}{36} = \frac{\sqrt{2}\pi+36\sqrt{2}}{72}$

$\approx \boxed{0.7688134885 \approx \cos(40°)}$

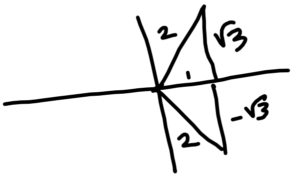
Actual: $\cos(40°) \approx$ 0.7660444431, to 10 decimal places.

8. Find all values of x such that $f(x) = \sin(2x) - x$ has a horizontal tangent on the interval…

   a. (5 pts) $[0, 2\pi)$

$f'(x) = 2\cos(2x) - 1 \overset{\text{SET}}{=} 0 \Rightarrow$

$\cos(2x) = \frac{1}{2}$



$\Rightarrow 2x = \frac{\pi}{3}, \frac{5\pi}{3}$. solves $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$

want all $x \in [0, 2\pi) \Rightarrow$

" " $2x \in [0, 4\pi)$

$2x = \frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$, $\frac{\pi}{3} + 4\pi$ is too big.

$2x = \frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$

This gives $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

   b. (5 pts) $(-\infty, \infty)$.

All the $2x$'s are $\frac{\pi}{3} + 2\pi n$, $\frac{5\pi}{3} + 2\pi n$.

" " x's are $\boxed{\frac{\pi}{6} + \pi n, \frac{5\pi}{6} + \pi n = x}$

9. (10 pts) Find $\frac{dy}{dx}$, given that $x^2y^2 + 2xy - \sin(xy) = 3$ $\Rightarrow$

$$2xy^2 + 2x^2yy' + 2y + 2xy' - (\cos(xy))(y + xy') = 0 \Rightarrow$$

$$2x^2yy' + 2xy' - x\cos(xy)y' = y\cos(xy) - 2xy^2 - 2y \Rightarrow$$

$$\boxed{y' = \frac{y\cos(xy) - 2xy^2 - 2y}{2x^2y + 2x - x\cos(xy)}}$$

SAME

B1 (5 pts) Find the derivative of $f(x)=\frac{1}{\sqrt{x}}$, by the definition of the derivative.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[f(x+h)-f(x)\right]$$

$$= \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] = \frac{1}{h}\left[\frac{1}{\sqrt{x+h}}\cdot\frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x+h}}{\sqrt{x+h}}\right]$$

$$= \frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right] = \frac{1}{h}\left[\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right]$$

$$= \frac{1}{h}\left[\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right] = \frac{1}{h}\left[\frac{-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right]$$

$$= \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \xrightarrow{h\to 0} \frac{-1}{x(2\sqrt{x})} = \boxed{\underbrace{\frac{-1}{2x\sqrt{x}}}_{f'(x)}} = -\frac{1}{2x^{3/2}}$$

$(h\neq 0)$

$$f(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}} \longrightarrow$$

$$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{1}{2}-\frac{2}{2}} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{-1}{2x^{3/2}} = \frac{-1}{2\sqrt{x^3}} = \frac{-1}{2\sqrt{x^2\cdot x}}$$

$$= \frac{-1}{2\sqrt{x^2}\sqrt{x}} = \frac{-1}{2|x|\sqrt{x}}$$. But

we already KNOW $x>0$, b/c $\mathcal{D}\left(\frac{1}{\sqrt{x}}\right) = (0,\infty)$ so $|x| = x$, here & we have $f'(x) = \frac{-1}{2x\sqrt{x}}$

B2 (5 pts) Prove that $\lim_{x\to 3}\left(x^2+5x+2\right)=26$

Let $f(x)=x^2+5x+2$, $L=26$.

Given $\varepsilon>0$, we want $\delta>0 \ni 0<|x-3|<\delta$ implies $|f(x)-L|<\varepsilon$.

We want $|x^2+5x+2-26| = |x^2+5x-24| = |x-3||x+8| < \delta|x+8|$.

We need a bound on $|x+8|$.

Assume $\delta \le 1$. Then $x\to 3$ & $|x-3|<\delta$

$\Rightarrow 2<x<4$

$10<x+8<12$, i.e.,

$|x+8|<12$

Now we can roll, with $\delta=\min\left\{1, \frac{\varepsilon}{12}\right\}$

Proof:

Let $\varepsilon>0$ be given. Define $\delta=\min\left\{1,\frac{\varepsilon}{12}\right\}$. Then

$0<|x-3|<\delta$ implies $|f(x)-L| = |x^2+5x+2-26| = |x^2+5x-24|$

$= |x-3||x+8| < \delta\cdot 12 \le \frac{\varepsilon}{12}\cdot 12 = \varepsilon$ ∎