1. (10 pts) Evaluate $\lim_{x \to 3} \frac{x^2 - 4}{x^3 - 8}$ by factoring.

x→2 was my intent. As is, x=3 is in the domain ⇒

$\lim_{x \to 3} \frac{x^2-4}{x^3-8} = \frac{3^2-4}{3^2-8} = \frac{5}{1} = 5$

$\lim_{x \to 2} \frac{x^2-4}{x^3-8} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} = \lim_{x \to 2} \frac{x+2}{x^2+2x+4} = \frac{4}{4+4+4} = \boxed{\frac{1}{3}}$

Bonus (5 pts) Prove that $\lim_{x \to 3}(x^2 - 2x + 1) = 4$.

Scratch: Want $|x^2-2x+1-4| < \varepsilon$

$\Rightarrow |x^2-2x-3| = |x-3||x+1|$

Need a bound on $|x+1|$ in the vicinity of $x=3$!

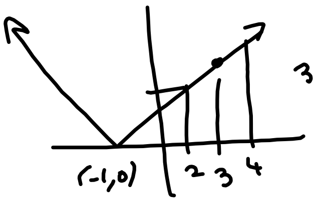
Say within 1 unit of 3, so $\delta \le 1$.

$\Rightarrow 2 < x < 4$

$+1 \quad +1 \quad +1$

$3 < x+1 < 5$

$\Rightarrow |x+1| < 5 \Rightarrow \frac{\varepsilon}{5}$



$3 < |x+1| < 5$ if $\delta \le 1$, i.e. $2 < x < 4$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{5}\}$.

Then $0 < |x-3| < \delta \Rightarrow |f(x) - L| = |x^2-2x+1-4| = |x^2-2x-3|$

$= |x-3||x+1| < 5|x-3| < 5\delta \le 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ∎

$$\lim_{x\to 2}(x^3-5x^2+1) = -11$$

Scratch $|x^3-5x^2+1-(-11)| = |x^3-5x^2+12|$

Divide by $x-2$:

2	1	-5	0	12
		2	-6	-12
	1	-3	-6	0

$x^3-5x^2+12 = (x-2)(x^2-3x-6)$

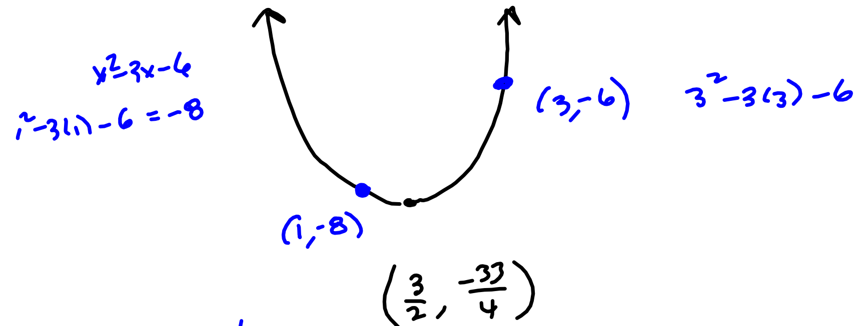
$|x^3-5x^2+12| = |x-2||x^2-3x-6|$

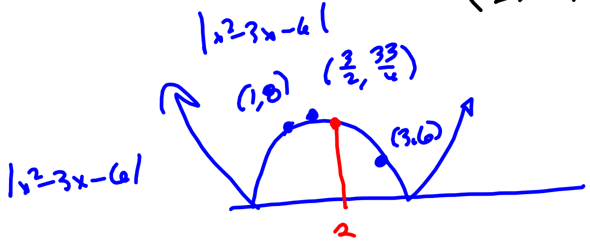
$x^2-3x-6 = x^2-3x+\left(\frac{3}{2}\right)^2-\frac{9}{4}-\frac{24}{4}$

$= \left(x-\frac{3}{2}\right)^2-\frac{33}{4}$

Looking at x^2-3x-6 in a nbhd of $x=2$

$\delta\le 1 \Rightarrow 1<x<3$





So $\delta\le 1 \Rightarrow |x^2-3x-6| < \frac{33}{4}$

$\frac{\varepsilon}{\frac{33}{4}} = \frac{4\varepsilon}{33}$

Proof

Let $\varepsilon>0$ be given. Define $\delta=\min\left\{1, \frac{4\varepsilon}{33}\right\}$

Then $0<|x-2|<\delta \Rightarrow |x^3-5x^2+1-(-11)| = |x^3-5x^2+12|$

$= |x-2||x^2-3x-6| < |x-2|\left|\frac{33}{4}\right| < \frac{33}{4}\delta \le \frac{33}{4}\cdot\frac{4\varepsilon}{33} = \varepsilon$ ∎

Two ways to think of linear approximations and differential approximations.

$y = L(x) = $ Tangent Line $=$ Linearization

$f(x) \approx L(x) = f'(x_1)\underbrace{(x-x_1)}_{\Delta x} + f(x_1)$ This is to approx. $f(x)$ near x_1

Differential Approach $f(x) \approx f'(x_1)\Delta x + f(x_1)$

Approximate $\sqrt{35}$

Linearization?

$f(x) = x^{\frac{1}{2}}$, $x_1 = 36$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(x_1) = \frac{1}{2\sqrt{36}} = \frac{1}{2(6)} = \frac{1}{12}$

$f(x_1) = \sqrt{36} = 6$

$L(x) = f'(x_1)(x-x_1) + f(x_1)$

$= \frac{1}{12}(x-36) + 6$

$f(35) \approx L(x) = \frac{1}{12}(35-36)+6$

$= \frac{1}{12}(-1) + 6$

$= \frac{72-1}{12} = \boxed{\frac{71}{12} \approx \sqrt{35}}$

Differential Way

$x_1 = 36$, $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$

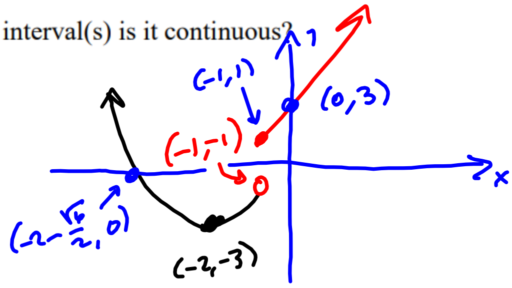
$\Delta x = x - x_1 = 35 - 36 = -1$

$f(35) \approx f'(36)\Delta x + f(36)$

$= \frac{1}{12}(-1) + 6$

$= 6 - \frac{1}{12} = \frac{71}{12} \approx \sqrt{35}$

5. (15 pts) Sketch the graph of the piecewise-defined function $f(x)=\begin{cases}2(x+2)^2-3 & \text{if } x<-1\\ 2x+3 & \text{if } x\geq -1\end{cases}$. On what interval(s) is it continuous?



Cnts on $(-\infty,-1)\cup(-1,\infty)$

$2(-1)+3=-2+3=1$

$2(x+2)^2-3=0$

$2(x+2)^2=3$

$(x+2)^2=\frac{3}{2}$

$x+2=\pm\sqrt{\frac{3}{2}\cdot\frac{2}{2}}$

$x=-2\pm\frac{\sqrt{6}}{2}$

Bonus (5 pts) What value of a will make $f(x)=\begin{cases}2(x+2)^2-3 & \text{if } x<-1\\ 2x+a & \text{if } x\geq -1\end{cases}$ continuous?

$a=1$ does it.

Brutal way:

Need $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(2(x+2)^2-3)=-1$

Want this to agree with

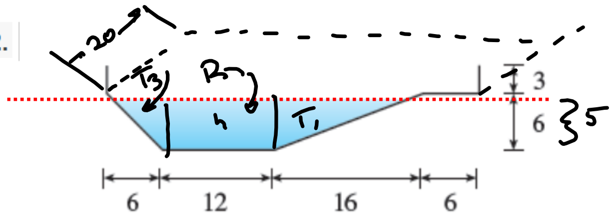
$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(2x+a) = 2(-1)+a \overset{\text{WANT}}{=} -1$

$a-2=-1$

$\boxed{a=1}$

A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at is deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of 0.6 ft^3/min, how fast is the water level rising when the depth at the deepest point is 5 ft? (Round your answer to five decimal places.)

12.



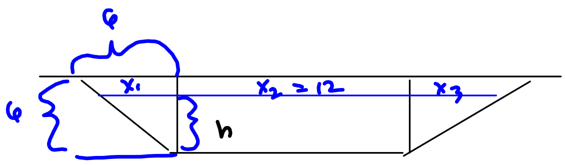
Section 2.8

Volume = cross-sectional area X width.

Let h = depth of water

Volume as function of h.

$\frac{6}{16}$



Area = Triangle 3 + Rectangle + Triangle 1

$= \frac{1}{2}(x_1)(h) + h \cdot 12 + \frac{1}{2}(x_3) h$

$\frac{x_1}{h} = \frac{6}{6} \Rightarrow x_1 = h$

$= \frac{1}{2}(h)(h) + 12h + \frac{1}{2}\left(\frac{8}{3}h\right)(h)$

$\frac{1}{2}h^2 + \frac{4}{3}h^2 + 12h$

$= \frac{3+2}{6}h^2 + 12h$

$= \frac{5}{6}h^2 + 12h$

$\frac{x_3}{h} = \frac{16}{6}$

$x_3 = \frac{8}{3}h$

$\Rightarrow$ Volume $= 20\left(\frac{5}{6}h^2 + 12h\right) = V$

$\frac{dV}{dt} = 20\left(\frac{5}{3}h \cdot \frac{dh}{dt} + 12\frac{dh}{dt}\right) = 20\left(\frac{5}{3}h + 12\right)\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{20\left(\frac{5}{3}h + 12\right)} = \frac{(.6)}{20\left(\frac{5}{3}(5) + 12\right)}$

$= \frac{.6}{20\left(\frac{25}{3} + \frac{36}{3}\right)} = \frac{\frac{6}{10}}{20\left(\frac{61}{3}\right)} = \frac{\frac{3}{5}}{20\left(\frac{61}{3}\right)} = \frac{\frac{3}{5} \cdot 3}{20(61)}$

$= \frac{\frac{9}{5}}{1220} = \frac{9}{5(1220)} =$

The figure is drawn without the top 3 feet. $V = \frac{1}{2}(b+12)h(20) = 10(b+12)h$ and, from similar triangles, $\dfrac{x}{h} = \dfrac{6}{6}$ and $\dfrac{y}{h} = \dfrac{16}{6} = \dfrac{8}{3}$, so $b = x + 12 + y = h + 12 + \frac{8h}{3} = 12 + \frac{11h}{3}$. Thus, $V = 10\left(24 + \frac{11h}{3}\right)h = 240h + \frac{110h^2}{3}$, and so $0.9 = \dfrac{dV}{dt} = \left(240 + \frac{220}{3}h\right)\frac{dh}{dt}$. When $h = 5$, $\frac{dh}{dt} = \frac{0.9}{240+5(220/3)} \approx 0.00148$ ft/min.