Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

8. $F = \frac{GmM}{r^2} = GmMr^{-2} \longrightarrow \frac{dF}{dr} = -2GmMr^{-1}$

Subtract 1 from -2
You get r-3, fool!

where G is the gravitational constant and r is the distance between the bodies.

(a) Find dF/dr.

(b) $\left.\frac{dF}{dr}\right|_{r=30000} = \cancel{-2}\,\frac{N}{Km} = -2GmMr^{-1} = \frac{\cancel{-2}\,GmM}{(30000)^3}$

$\longrightarrow 30000^3 = GmM$

want $\left.\frac{dF}{dr}\right|_{r=15000} = -2GMm\left(\frac{1}{15000^3}\right)$

$= -2(30000)^3\left(\frac{1}{15000^3}\right) = -4\,\frac{N}{Km}$

$= -2(2^3) = -16\ N/km$

(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when r = 30,000 km. How fast does this force change when r = 15,000 km?

$GmM\frac{d}{dr}\left[\frac{1}{r^2}\right] = GMm\frac{d}{dr}\left[r^{-2}\right] = GMm\left[-2r^{-3}\right]$!

Section 2.9 - Differentials

$\frac{dy}{dx} = f'(x)$

The differential of y is

$dy = f'(x)dx$

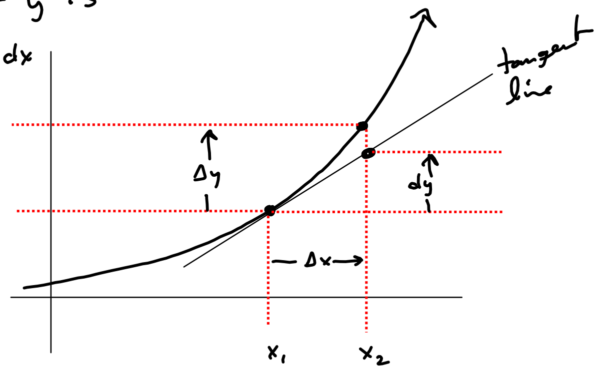
$\frac{\Delta y}{\Delta x} \approx f'(x)$

$\Delta y \approx f'(x)\Delta x$

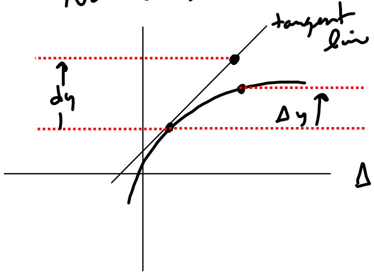
$f(x_2) - f(x_1) \approx f'(x)\Delta x$

$f'(x)\Delta x = f'(x)dx$

$\Delta x = dx$



Notice that in this picture $\Delta y > dy$ (concave up)



$\Delta y < dy$ here, b/c tangent line is ABOVE $f(x)$.

Sometimes the differential is simpler/less work than the actual calculation.

Use a differential to estimate the volume of paint needed to coat a sphere of radius 10 meters with a coat that is 0.1 cm thick.

The trick is that the *newly painted sphere* has a radius that is 0.1 cm greater than the unpainted sphere. We find the difference in volume directly:

$V = \frac{4}{3}\pi r^3$ = Volume of sphere

$\Delta V = V(10\text{m} + .1\text{cm}) - V(10\text{m})$

$\left((.1\text{ cm})\left(\frac{1\text{ m}}{100\text{ cm}}\right) = .001\text{ m} \right)$

$= V(10.001) - V(10)$

$= \frac{4}{3}\pi(10.001)^3 - \frac{4}{3}\pi(10)^3 \approx \boxed{1.256763\text{ m}^3}$

$V := r \rightarrow \frac{4}{3} \cdot \text{Pi} \cdot r^3$

$$V := r \mapsto \frac{4 \cdot \pi \cdot r^3}{3}$$

$V(10.001) - V(10)$

$$1.256763$$

Now, use a differential

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

$\Rightarrow dV = 4\pi r^2 dr$

$dr = \Delta r = .001\text{ m}$

$r = 10$

$dV = 4\pi(10)^2(.001) = 4\pi(.1)$

$V := r \rightarrow \frac{4}{3} \cdot \mathrm{Pi} \cdot r^3$

$V := r \mapsto \frac{4 \cdot \pi \cdot r^3}{3}$

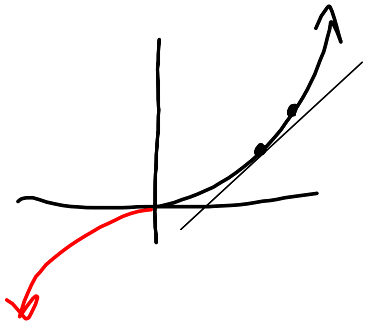
$V(10.001) - V(10)$

$\Delta V \approx 1.256763$ Direct m^3

$4 \cdot \mathrm{Pi} \cdot .1$

$dV = 4\pi r^2 dr \approx 1.256637062 \ m^3$

$\frac{4\pi r^3}{3}$



The differential is easier if you're doing this by hand or with a scientific calculator, generally.

Notice that the differential gives us a small underestimate. This is because the slope of the curve is growing, but the slope of the tangent line is constant and less than the slop to the right of $r = 10$.

Estimate $\sin(42^{\circ})$ with a tangent line.

Note that we <u>know</u> that $\sin(45^{\circ}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

That's where we build the tangent line.

<u>NOTE</u> THIS <u>only</u> works in radians.

Tangent Line for $\sin x$ @ $x = \frac{\pi}{4}$:

$$\frac{d}{dx}[\sin(x)] = \cos(x) = f'(x) \qquad \boxed{x_1 = \frac{\pi}{4}}$$

$$f'(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$y = f'(x_1)(x - x_1) + f(x_1) = \cos(\tfrac{\pi}{4})(x - \tfrac{\pi}{4}) + \sin(\tfrac{\pi}{4})$$

$$= \frac{\sqrt{2}}{2}(x - \tfrac{\pi}{4}) + \frac{\sqrt{2}}{2}$$

$$42^{\circ} = 45^{\circ} - 3^{\circ}$$

$$\Delta x = -3^{\circ}$$

CONVERT TO RADIANS!

$(-3^{\circ})(\frac{\pi}{180^{\circ}}) = \frac{-\pi}{60}$ This will replace the $(x - x_1)$

$$(42^{\circ})(\frac{\pi}{180^{\circ}}) = \frac{42}{180}\pi = \frac{7\pi}{30}$$

$$y = \frac{\sqrt{2}}{2}\left(\frac{7\pi}{30} - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} \qquad \frac{7\pi}{30} - \frac{\pi}{4} = \frac{14\pi - 15\pi}{60} = \frac{-\pi}{60}$$

$$= \frac{\sqrt{2}}{2}(-\tfrac{\pi}{60}) + \frac{\sqrt{2}}{2}$$

$$\approx -\frac{\sqrt{2}\pi}{120} + \frac{\sqrt{2}}{2}\cdot\frac{60}{60} = \frac{-\sqrt{2}\pi + \sqrt{2}\cdot 60}{120}$$

2|30
3|15
5

$$-\frac{\text{sqrt}(2)\cdot\text{Pi}}{120}+\frac{\text{sqrt}(2)}{2}$$

$$-\frac{\sqrt{2}\,\pi}{120}+\frac{\sqrt{2}}{2}$$

$evalf(\%)$

$$0.6700827565$$

$$\sin\left(\frac{42\cdot\text{Pi}}{180}\right)$$

$$\sin\left(\frac{7\,\pi}{30}\right)$$

$evalf(\%)$

$$0.6691306063$$

I love these kinds of questions. Other variations:

$\sqrt{97}$

$x_1=100,\ f(x)=\sqrt{x}=x^{\frac{1}{2}} \longrightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

$f(100)=\sqrt{100}=10$

$f'(100)=\frac{1}{2(10)}=\frac{1}{20}$

$y=f'(x_1)(x-x_1)+f(x_1)$

$=\frac{1}{20}(97-100)+10$

$=\frac{-3}{20}+10=\frac{-3+200}{20}=\frac{197}{20}$

$f := x \rightarrow \mathrm{sqrt}(x)$

$$f := x \mapsto \sqrt{x}$$

$fp := \mathrm{D}(f)$

$$fp := x \mapsto \frac{1}{2 \cdot \sqrt{x}}$$

$y := x \rightarrow fp(100) \cdot (x - 100) + f(100)$

$$y := x \mapsto fp(100) \cdot (x - 100) + f(100)$$

$y(97)$

$$-\frac{3\sqrt{100}}{200} + 10$$

$simplify(\%)$

$$\frac{197}{20}$$

$evalf(\%)$

$$9.850000000$$

$evalf(\mathrm{sqrt}(97))$

$$9.848857802$$