Recall - Composition of functions $y = (f \circ u)(x) = f(u(x)) = f$ composed with u of x. A function of a function.

Example: $y = h(x) = (3x-2)^2$

Let $u(x) = 3x-2$ and $f(u) = u^2$

Then $h(x) = f(u(x)) = u(x)^2 = (3x-2)^2$

We have a trick for taking the derivatives of such composite functions that is very useful:

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

By this, $h'(x) = \left(f(u(x))\right)' = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

$\frac{df}{du} = \frac{d}{du}[u^2] = 2u$, $\frac{du}{dx} = 3 \Longrightarrow$

for comparison:

$\frac{dh}{dx} = \frac{dh}{du} \cdot \frac{du}{dx} = 2u \cdot 3 = 6u = \boxed{6(3x-2)} = 18x - 12$

Old way

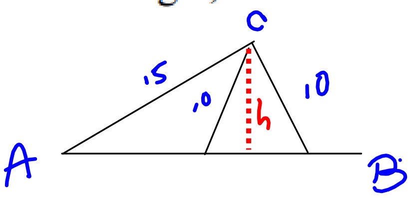
$h(x) = (3x-2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4$

$\Longrightarrow h'(x) = 18x - 12$ ✓

Questions about Writing Project #1 or #0?

6. Suppose $A = 33^0$, $a = 10$ cm, and $b = 15$ cm.

   a. (5 pts) Show that there are two solutions to this triangle, before solving the triangle.



Need $a < b$ & $a > h$

$10 < 15$ ✓

$h = 15 \sin 33^0 \approx$ 8.169585525 < 10

$\frac{h}{15} = \sin 33^0$

$h = 15 \sin(33^0)$

So $\boxed{h < a}$

$h \approx 8.17 < a = 10 < 15 = b.$

b. (5 pts) (Law of Sines) For one, B will be acute. That's the first solution that the Law of Sines will produce. For the other, B will be obtuse. Round final answers to 3 decimal places. See figure on the right.

Assume $A = 27^0$, $a = 10$, $b = 12$

Law of Sines

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a} = \frac{12 \sin 27^0}{10} \approx .5447885997$$

```
12sin(27)/10
          .5447885997
sin⁻¹(Ans
          33.01021846
180-27-Ans
          119.9897815
```

$\Rightarrow B \approx 33.01021846 \approx \boxed{33.010^0 \approx B}$

$\Rightarrow C = 180^0 - A - B$

$\approx 180^0 - 27^0 - 33.\ldots^0$

$\approx 119.9897815 \approx \boxed{C \approx 119.990^0}$

Little c:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a \sin C}{\sin A} = \frac{10 \sin(119.\ldots^0)}{\sin(27^0)} \approx \boxed{19.078 \text{ cm} \approx c}$$

```
sin⁻¹(Ans
          33.01021846
180-27-Ans
          119.9897815
10sin(Ans)/sin(
          19.0778125
```

Now, for B obtuse, call it $\hat{B}$, we subtract from 180°

$$\hat{B} = 180^\circ - B \approx 33.01021846^\circ \approx 146.9897815 \approx \boxed{\hat{B} \approx 146.990^\circ.}$$

```
180-27-Ans
     119.9897815
10sin(Ans)/sin(▸
      19.0778125
180-33.01021846
     146.9897815
█
```

S2.3 questions

Differentiate.

9. $F(y) = \left(\frac{1}{y^2} - \frac{9}{y^4}\right)(y + 3y^3)$ $= \underbrace{(y^{-2} - 9y^{-4})}_{f}\underbrace{(y+3y^3)}_{g}$

$\Rightarrow F'(y) = (fg)' = f'g + fg'$

$= \boxed{(-2y^{-3} + 36y^{-5})(y+3y^3) + (y^{-2} - 9y^{-4})(1+9y^2)}$

Stop Right There!

#13 $\frac{w}{w+\frac{2}{w}} = \frac{w}{w+2w^{-1}} = \frac{f}{g} \Rightarrow \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} =$

$= \frac{1(w+2w^{-1}) - w(1-2w^{-2})}{(w+2w^{-1})^2}$

Some 2.4 Examples

Differentiate the following:

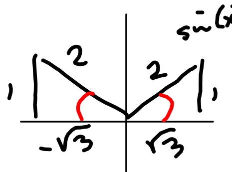
$$y = \frac{x\sin(x)}{x+\cos(x)} \Longrightarrow \frac{(1\sin(x) + x\cos(x))(x+\cos(x)) - (x\sin(x))(1-\sin(x))}{(x+\cos(x))^2}$$

$f = x\sin(x) \Longrightarrow f' = 1\sin(x) + x\cos(x)$ (PRODUCT RULE)

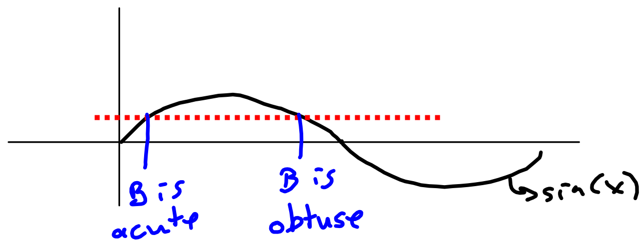
$g = x+\cos(x) \Longrightarrow g' = 1-\sin(x)$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$\sin(x) = \frac{1}{2}$



$\sqrt{2^2 - 1^2} = \sqrt{3}$



$\arcsin\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30^\circ = \frac{\pi}{6}$

$\sin(x) = \frac{1}{2}$ has 2 sol'ns on $[0, 2\pi)$

$x = \frac{\pi}{6}, \quad x = \pi - \frac{\pi}{6}$

$x = 30^\circ, \quad x = 180^\circ - 30^\circ$

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$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

EXAMPLE Let $h(x) = (f \circ u)(x) = f(u(x)) = (3x-2)^2$

Then, by chain rule, with $f(u) = u^2$ and $u(x) = 3x-2$

$$\frac{df}{du} \cdot \frac{du}{dx} = 2u \cdot 3 = 6u = 6(3x-2) = 18x-12$$

Confirm:

$$(3x-2)^2 = (3x)^2 - 2(3x)(2) + (2)^2$$

$$= 9x^2 - 12x + 4$$

$$\Rightarrow h'(x) = 18x-12$$

Proof

Let $y=f(u(x))$, where u is difbl at x and f is difbl at $u(x)$.

Define $\Delta u = u(x+\Delta x) - u(x)$

& $\varepsilon_1 = \varepsilon_1(x) = \frac{\Delta u}{\Delta x} - u'(x) \qquad (\Delta x \neq 0)$

If u is difbl, then ε_1 is small when Δx is small

$\Rightarrow \boxed{\varepsilon_1 \Delta x + u'(x)\Delta x = \Delta u} = (\varepsilon_1 + u'(x))\Delta x$

Define $\Delta y = f(u+\Delta u) - f(u)$

& $\varepsilon_2 = \frac{\Delta y}{\Delta u} - f'(u)$

$\Rightarrow \boxed{\varepsilon_2 \Delta u + f'(u)\Delta u = \Delta y} = (\varepsilon_2 + f'(u))\Delta u$, i.e

$\Delta y = (f'(u)+\varepsilon_2)(u'(x)\Delta x + \varepsilon_1)\Delta x$

$\Rightarrow \frac{\Delta y}{\Delta x} = (f'(u)+\varepsilon_2)(u'(x) + \varepsilon_1)$

Now, $\Delta x \to 0$ implies $\varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$
(u difbl @ x) (f difbl @ u)

$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = h'(x) = f'(u(x))\,u'(x)$ ■