We left off proving

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Want to prove that

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\cos(x) = \sin\left(\tfrac{\pi}{2} - x\right) = \sin\left(\tfrac{\pi}{2}\right)\cos(-x) + \sin(-x)\cos\left(\tfrac{\pi}{2}\right)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h) - \cos(x)}{h} = \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \cos(x)\left(\frac{\cos(h)-1}{h}\right) - \frac{\sin(x)\sin(h)}{h} =$$

$$= \cos(x)\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h} \xrightarrow{h \to 0} \cos(x)\cdot 0 - \sin(x)\cdot 1$$

$$= -\sin(x) \quad \blacksquare$$

$$\boxed{\frac{d}{dx}[\cos(x)] = -\sin(x)}$$

$$f = \sin(x) \Rightarrow f' = \cos(x)$$
$$g = \cos(x) \Rightarrow g' = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{d}{dx}\left[\frac{f}{g}\right] = \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \qquad \frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

We're ahead of schedule on lecture. I'd like to see if we can do some midterm prep.

Some Old Written Tests

Fall '17 tests feature absolute value in the limit

Let $f(x) = \frac{2x^3+13+21}{|x+3|}$

4. (5 pts each) Evaluate the following limits, if they exist. If one does not exist, explain why.

a. $\lim\limits_{x\to -3^+} \frac{2x^2+13x+21}{|x+3|} = \lim\limits_{x\to -3^+} f(x)$ b. $\lim\limits_{x\to -3^-} \frac{2x^2+13x+21}{|x+3|} = \lim\limits_{x\to -3^-} f(x)$ c. $\lim\limits_{x\to -3} \frac{2x^2+13x+21}{|x+3|}$

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases} = \begin{cases} x+3 & \text{if } x \geq -3 \\ -(x+3) & \text{if } x < -3 \end{cases}$$

a) $\lim\limits_{x\to -3^+} \frac{2x^2+13x+21}{|x+3|} = \lim\limits_{x\to -3^+} \frac{2x^2+13x+21}{x+3} = \lim\limits_{x\to -3^+} \frac{(2x+7)(x+3)}{x+3}$

Scratch:

$$\begin{array}{r|rrr} -3 & 2 & 13 & 21 \\ & & -6 & -21 \\ \hline & 2 & 7 & 0 \\ & x' & c & r \end{array}$$

$= \lim\limits_{x\to -3^+} (2x+7) = 2(-3)+7 = \boxed{1 = \lim\limits_{x\to -3^+} f(x)}$

b) $\lim\limits_{x\to -3^-} f(x) = \lim\limits_{x\to -3^-} \frac{(2x+7)(x+3)}{-(x+3)} = \lim\limits_{x\to -3^-} \left(\frac{2x+7}{-1}\right) = \frac{1}{-1} = \boxed{-1 = \lim\limits_{x\to -3^-} f(x)}$

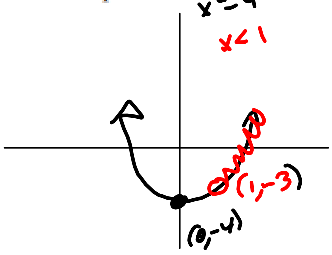
c) $\lim\limits_{x\to 3} \frac{2x^2+13x+21}{|x+3|}$ $\nexists$ b/c $\lim\limits_{x\to -3^+} f(x) = 1 \neq -1 = \lim\limits_{x\to -3^-} f(x)$ ■

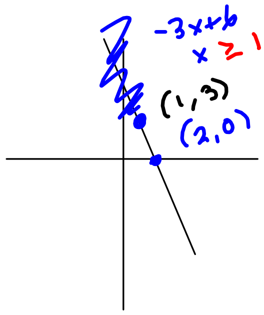
Does not exist because the limit from the right isn't the same as the limit from the left.

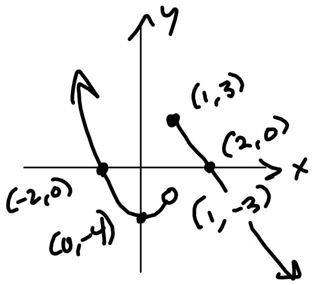
5. (15 pts) Sketch the graph of the piecewise-defined function $f(x)=\begin{cases} x^2-4 & \text{if } x<1 \\ -3x+6 & \text{if } x\leq 1 \end{cases}$. Label intercepts with ordered pairs.

$\geq$!!

$x=1$ is "suture point"





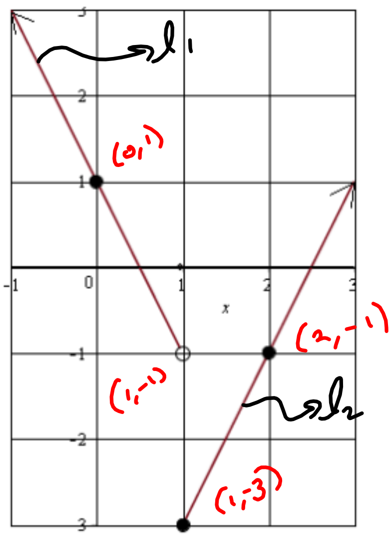


$x^2-4=0$

$x^2=4$

$x=\pm 2$

4. (5 pts) Write the definition of the piecewise-defined function from its graph:



l_1: $(0,1), (1,-1)$

$m_1 = \frac{-1-1}{1-0} = -2$

$y = m_1(x-x_1)+y_1$

$y = -2(x-0)+1$

l_2: $(1,-3), (2,-1)$

$m_2 = \frac{-1-(-3)}{2-1} = \frac{-1+3}{1} = \frac{2}{1} = 2$

$y = 2(x-1)-3$

$$f(x) = \begin{cases} -2x+1 & \text{if } x<1 \\ 2(x-1)-3 & \text{if } x \geq 1 \end{cases}$$

where is it cont's?

$(-\infty, 1) \cup [1, \infty) \longrightarrow \boxed{(-\infty, 1) \cup (1, \infty)}$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

$x^3 - y^3 = (x-y)(x^2+xy+y^2)$

$x^3+y^3 = (x+y)(x^2-xy+y^2$

Differentiate $\sqrt[3]{x}$ by the limit definition of the derivative.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} = \frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\left(\sqrt[3]{x+h}^2+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x}^2\right)}{h\left(\sqrt[3]{x+h}^2+\sqrt[3]{x(x+h)}+\sqrt[3]{x}^2\right)}$$

$$= \frac{x+h-x}{h\left(\sqrt[3]{x+h}^2+\sqrt[3]{x(x+h)}+\sqrt[3]{x}^2\right)}$$

$$= \frac{\cancel{h}}{\cancel{h}\left(\sqrt[3]{x+h}^2+\sqrt[3]{x(x+h)}+\sqrt[3]{x}^2\right.} \xrightarrow{h\to 0} \frac{1}{\sqrt[3]{x}^2+\sqrt[3]{x^2}+\sqrt[3]{x^2}}$$

$$\sqrt[a]{x^b} = \left(\sqrt[a]{x}\right)^b \qquad = \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3x^{2/3}}$$

Alternate

$$f(x) = \sqrt[3]{x}$$

$$f'(a) = \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}$$

Scratch

$$\left(\begin{aligned} x - a &= \left(x^{\frac{1}{3}}\right)^3 - \left(a^{\frac{1}{3}}\right)^3 \\ &= \left(x^{\frac{1}{3}} - a^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}} + (ax)^{\frac{1}{3}} + a^{\frac{2}{3}}\right) \end{aligned} \right)$$

$$= \frac{x^{\frac{1}{3}} - a^{\frac{1}{3}}}{\left(x^{\frac{1}{3}} - a^{\frac{1}{3}}\right)\left(x^{2/3} + (ax)^{\frac{1}{3}} + a^{2/3}\right)} \underset{a \neq x}{=} \frac{1}{\left(x^{2/3} + (ax)^{\frac{1}{3}} + a^{\frac{2}{3}}\right)} \xrightarrow{x \to a}$$

$$\frac{1}{a^{\frac{2}{3}} + a^{2/3} + a^{\frac{2}{3}}} = \frac{1}{3a^{\frac{2}{3}}}$$

Either way you do this one in Section 2.2, the trick is recognizing the difference of cubes living inside the thing.

Define

$$\Delta u = g(a+\Delta x) - g(a)$$

& $\varepsilon_1 = \frac{\Delta u}{dx} - g'(a)$

Then $\Delta u = \left(g'(a) + \varepsilon_1\right)\Delta x$

Define

$$\Delta y = f(g(a+\Delta v)) - f(g(a))$$

& $\varepsilon_2 = \frac{\Delta y}{\Delta g} - f'(g(a))$

Then $\Delta y = \left(f'(g(a)) + \varepsilon_2\right)\Delta u$

$\therefore$ & so $\Delta y = \left(f'(g(a)) + \varepsilon_2\right)\left(g'(a) + \varepsilon_1\right)dx$

This is from Section 2.5, before you have a heart attack.

Chain Rule