Section 2.3 - Differentiation Formulas

Let b, a be constants and f, g be differentiable functions

Denote $f'(x)$ by f', $g'(x)$ by g'

Then:

$\frac{d}{dx}[b] = 0$ (horizontal line $y=7$, e.g.)

$\frac{d}{dx}[x^n] = nx^{n-1}$ $\forall\, n \neq 0$. (For $n=0$, see above (constant!))

So $\frac{d}{dx}[5x^7 - 3x^5 + 4x^3 + x^2 + 2] = 35x^6 - 15x^4 + 12x^2 + 2x = f'(x)$

$\frac{d}{dx}[af + bg] = a\frac{df}{dx} + b\frac{dg}{dx}$

$(af+bg)' = af' + bg'$

Product Rule: $(fg)' = f'g + fg'$

$\boxed{\text{Pf}}$ $(fg)' = \lim_{h\to 0} \frac{(f(x+h)g(x+h)) - f(x)g(x)}{h}$

Now $\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$

$= \frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h}$

$= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}$

$\xrightarrow{h\to 0} f'(x)g(x) + f(x)g'(x)$ ∎

$\boxed{(fg)' = f'g + fg' \quad \text{PRODUCT RULE}}$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Pf

$$\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right] = \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{(f(x+h) - f(x))g(x)}{g(x)g(x+h)} + \frac{f(x)(g(x) - g(x+h))}{g(x)g(x+h)}\right]$$

$$= \frac{1}{h}\left[\frac{f(x+h) - f(x)}{g(x)g(x+h)}g(x) - \frac{f(x)(g(x+h) - g(x))}{g(x)g(x+h)}\right]$$

$$= \frac{f(x+h) - f(x)}{h} \cdot \frac{g(x)}{g(x)g(x+h)} - f(x)\frac{g(x+h) - g(x)}{h} \cdot \frac{1}{g(x)g(x+h)}$$

$$= f'(x) \cdot \frac{g(x)}{g(x)^2} - f(x)g'(x)\left(\frac{1}{g(x)^2}\right)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \boxed{\frac{f'g - fg'}{g^2} = \left(\frac{f}{g}\right)'}$$

QUOTIENT RULE

22. [-/1 Points] DETAILS SCALC8 2.3.072.

If $h(2) = 5$ and $h'(2) = -7$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Bigg|_{x=2.} = k'(2)$$

§2.3 #22

If $K(x) = \frac{h(x)}{x}$, then what's $k'(2)$, given $h(2)=5$ & $h'(2)=-7$

$f(x) = h(x)$
$g(x) = x$

$$\Longrightarrow \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{h'(x)x - h(x)\cdot 1}{x^2} = k'(x) \Longrightarrow k'(2) =$$

$$= \frac{-7(2) - 5(1)}{2^2} = \frac{-14-5}{4} = \frac{-19}{4} = -4.75$$

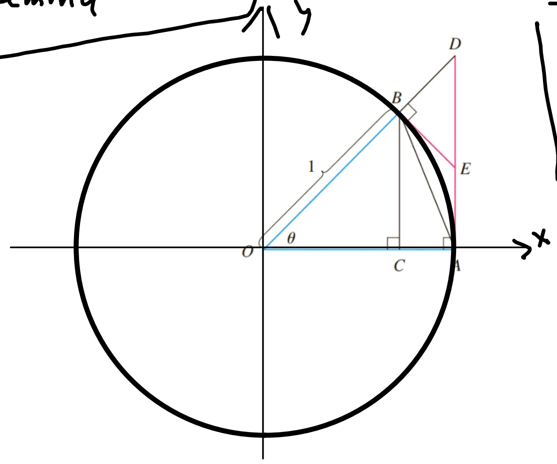
§ $h(2)=9$ & $h'(2)=-7 \Longrightarrow$

$$\frac{-7(2) - 9(1)}{2^2} = \frac{-14-9}{4} = -\frac{23}{4}$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Lemma

Pf: Consider a small sector of the unit circle in 1^{st} Quadrant.

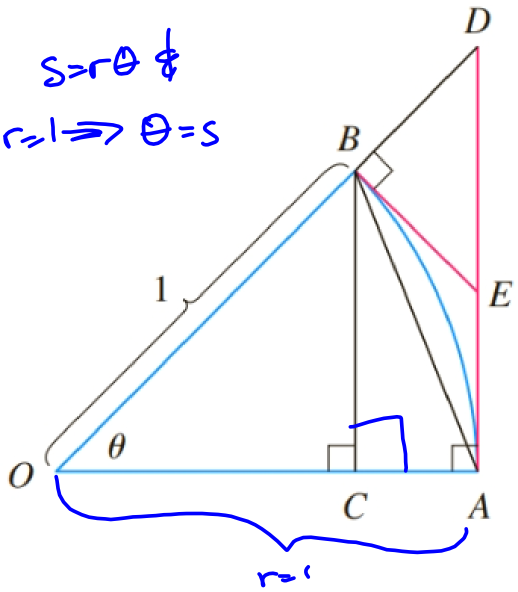


Want to prove

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$s = r\theta$ &

$r = 1 \Rightarrow \theta = s$



On the unit circle,

Arc Length (Blue)

$s = \theta$.

(i.) $\boxed{\frac{\sin\theta}{\theta} < 1}$

(ii.) $\boxed{\frac{\sin\theta}{\theta} > \cos\theta}$

$\cos\theta < \frac{\sin\theta}{\theta} < 1$

$\theta \to 0$

$1 \le \lim \frac{\sin\theta}{\theta} \le 1$

(iii.) Squeeze Theorem.

(i) To prove this, we prove that $\theta > \sin\theta$ =

$\sin\theta = \frac{|BC|}{1} = |BC| < |AB| < arc(AB) = \theta$

$\sin\theta < \theta \Longrightarrow$

$\boxed{\frac{\sin\theta}{\theta} < 1}$ for $\theta > 0$

(ii) We prove that $\theta < \tan\theta$

$\theta = arc(AB) < |AE| + |EB| < |AE| + |ED| = |AD|$

$= \frac{|AD|}{1} = \tan\theta$, i.e., $\theta < \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\longrightarrow \cos\theta < \frac{\sin\theta}{\theta}$

Combine:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$\theta \to 0$ $\quad$ $\theta \to 0$ $\quad$ $\theta \to 0$

$$1 \le \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \le 1$$

$$\therefore \boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$ ✓

Claim: $\frac{d}{dx}[\sin(x)] = \cos(x)$

Proof $\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin(x)}{h}$

$= \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$

$= \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\sin(h)\cos(x)}{h}$

$= \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \frac{\sin(h)}{h}\cos(x)$

$\xrightarrow{h\to 0} \lim_{h\to 0} \sin(x)\left(\frac{\cos(h)-1}{h}\right) + 1\cos(x)$

Need $\frac{\cos(h)-1}{h} \xrightarrow{h\to 0} 0$ & we're done

$\left(\frac{\cos(h)-1}{h}\right)\left(\frac{\cos(h)+1}{\cos(h)+1}\right) = \frac{\cos^2(h)-1}{h(\cos(h)+1)} = \frac{-(1-\cos^2(h))}{h(\cos(h)+1)}$

$= \frac{-\sin^2(h)}{h}\left(\frac{1}{\cos(h)+1}\right) = \frac{-\sin(h)}{h}\left(\frac{\sin(h)}{\cos(h)+1}\right)$

$\xrightarrow{h\to 0} -1\left(\frac{0}{2}\right) = 0 \longrightarrow$

$\boxed{\frac{d}{dx}[\sin(x)] = \cos(x)}$ ☒

Nathan asks 2.1#5

5. (a) Find the slope m of the tangent to the curve $y = \frac{3}{\sqrt{x}}$ at the point where $x = a > 0$.

$m =$ [ ]

Nathan's version

$y = \frac{3}{\sqrt{x}}$

$\mathcal{D}$: Need $\sqrt{x} \neq 0$ & $x \geq 0$ } $x > 0$

$\mathcal{D} = (0, \infty)$

(b) Find equations of the tangent line at the point (1, 7).

$\frac{3}{\sqrt{1}} = 3$

$y =$ [ ]

Find equations of the tangent line at the point $\left(4, \frac{7}{2}\right)$.

$\frac{3}{2}$

$y(4) = \frac{3}{\sqrt{4}} = \frac{3}{2}$

$y =$ [ ]

(c) Graph the curve and both tangents on a common screen.

$$\frac{1}{h}\left[\frac{3}{\sqrt{x+h}} - \frac{3}{\sqrt{x}}\right] = \frac{1}{h}\left[\frac{3\sqrt{x} - 3\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right]\left[\frac{3\sqrt{x}+3\sqrt{x+h}}{3\sqrt{x}+3\sqrt{x+h}}\right]$$

$$= \frac{1}{h}\left[\frac{9x - 9(x+h)}{\sim}\right] = \frac{1}{h}\left[\frac{9x - 9x - 9h}{\sim}\right]$$

$$= \frac{-9h}{h[\sim]} = \frac{-9}{\sim} = \frac{-9}{\sqrt{x}\sqrt{x+h}\,(3\sqrt{x}+3\sqrt{x+h})} \xrightarrow{h \to 0} \frac{-9}{\sqrt{x}\sqrt{x}\,(3\sqrt{x}+3\sqrt{x})} = \frac{-9}{6\sqrt{x}\cdot x} = \frac{-3}{2\sqrt{x}\cdot x} = \frac{-3}{2x^{3/2}}$$

Check (cheat):

$$\frac{3}{\sqrt{x}} = 3x^{-\frac{1}{2}} \Rightarrow f'(x) = -\frac{3}{2}\left(x^{-\frac{1}{2}-1}\right) = \frac{-3}{2x^{3/2}}$$