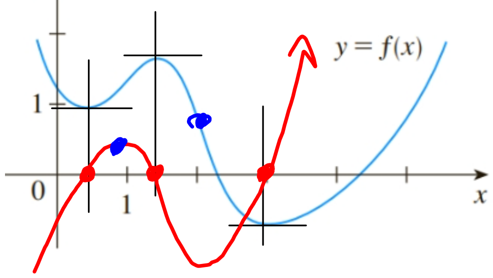
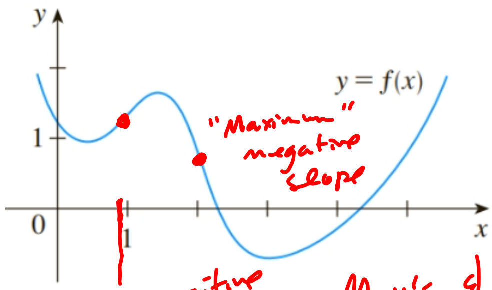
Section 2.2 - The Derivative as a Function

$$\boxed{2} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative gives us the slope of *f* at every point for which the above limit is defined.

Example 1: We try to graph *f '* given the graph of *f* :



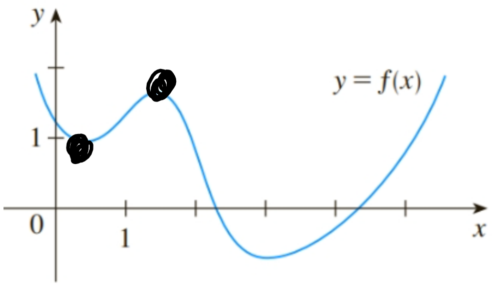


"Maximum" negative slope

Max positive slope

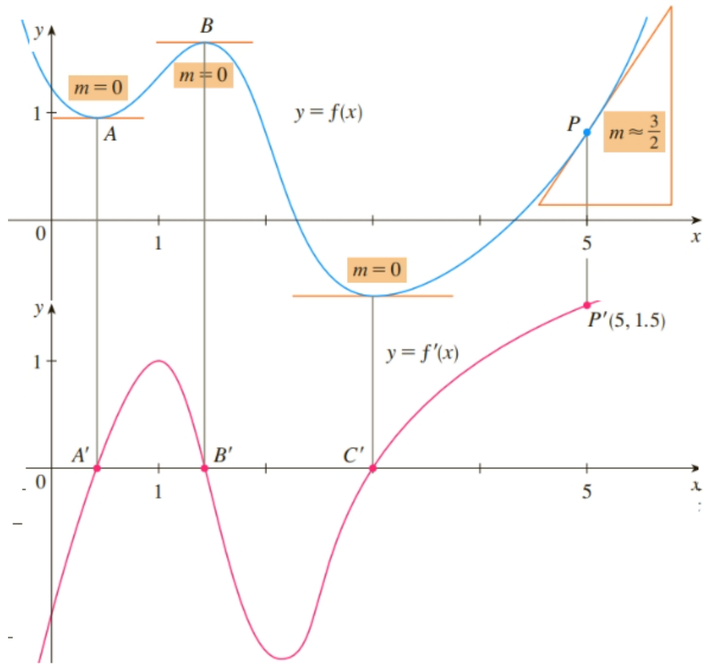
Max's & min's of the derivative are "inflection points."

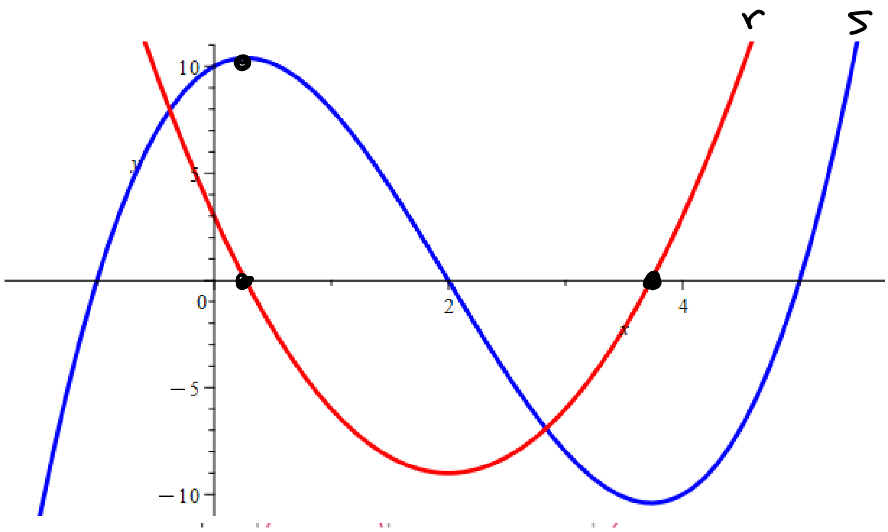
Derivative of the derivative is zero, just like zeros of the derivative are max's/min's of f(x).



$f'(x)$ = slope of f

$f''(x)$ = slope of f'


y
B
m = 0
m = 0
1
A
y = f(x)
P
m ≈ 3/2
0
1
5
x
m = 0
y
P′(5, 1.5)
y = f′(x)
1
A′
B′
C′
0
1
5
x



Which is *f* and which is *f ′* ?

A huge percentage of applications of differential calculus consist of optimization.

Optimum profit - maximum: f ' = 0.

Minimize pollution - minimim: f ' = 0.

"Take the derivative, set it equal to zero and solve."

Typically, the derivative exists for all of the domain of a given function. What you need to look for is places where *f* might come to a "point" or "cusp."

$f(x) = x^{\frac{2}{3}}$ $\quad \mathcal{D} = (-\infty, \infty) = \mathbb{R}$

$$\frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h} = \frac{\left((x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}\right)\left(\left((x+h)^{\frac{2}{3}}\right)^2 + (x+h)^{\frac{2}{3}}(x)^{\frac{2}{3}} + \left(x^{\frac{2}{3}}\right)^2\right)}{h\left((x+h)^{4/3} + \left(x(x+h)\right)^{2/3} + x^{\frac{4}{3}}\right)}$$

$$= \frac{\left((x+h)^{\frac{2}{3}}\right)^3 - \left(x^{\frac{2}{3}}\right)^3}{\sim\sim} = \frac{(x+h)^2 - x^2}{\sim\sim} = \frac{x^2 + 2xh + h^2 - x^2}{\sim\sim}$$

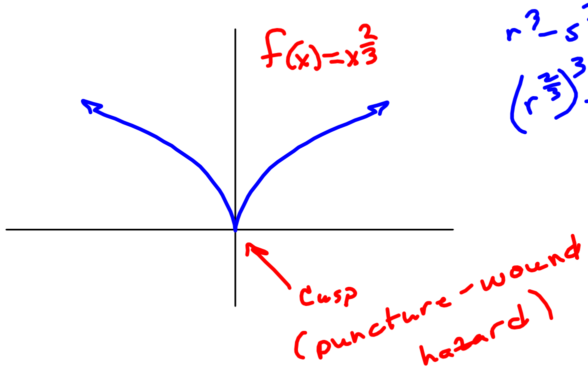
$$= \frac{2xh + h^2}{\sim\sim} = \frac{\cancel{h}(2x+h)}{\cancel{h}\left((x+h)^{4/3} + \left(x(x+h)\right)^{\frac{2}{3}} + x^{\frac{4}{3}}\right)} \xrightarrow{h \to 0}$$

$$\frac{2x}{x^{\frac{4}{3}} + x^{\frac{4}{3}} + x^{\frac{4}{3}}} = \frac{2x}{3x^{\frac{4}{3}}} = \frac{2}{3x^{1/3}} \quad \mathcal{D} = (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$$

$$r^3 - s^3 = (r-s)(r^2 + rs + s^2)$$

$$\left(r^{\frac{2}{3}}\right)^3 - \left(s^{\frac{2}{3}}\right)^3 = \left(r^{\frac{2}{3}} - s^{\frac{2}{3}}\right)\left(\left(r^{2/3}\right)^2 + r^{\frac{2}{3}}s^{\frac{2}{3}} + \left(s^{\frac{2}{3}}\right)^2\right)$$

$$\left((x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}\right)\Big(\quad \uparrow \quad \Big)$$



$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -\infty$$

$$\frac{2}{3x^{1/3}} = f'(x)$$

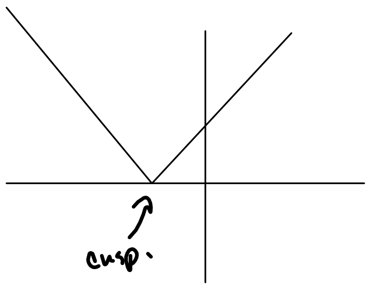
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = +\infty$$

In any case $f'(0) \nexists$

Book example:

$$f(x) = |x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases}$$

$$= \begin{cases} x+3 & \text{if } x \geq -3 \\ -(x+3) & \text{if } x < -3 \end{cases}$$



$$f'(x) = \begin{cases} 1 & \text{if } x > -3 \\ -1 & \text{if } x < -3 \end{cases}$$

But we need to take the limit from BOTH directions @ $x=-3$

$$\lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} =$$

$$\lim_{h \to 0^-} \frac{-(-3+h+3) - (-(-3+3))}{h} =$$

$|x+3|$

$$\frac{3-h-3+0}{h} = \frac{-h}{h} \xrightarrow{h \to 0} -1$$

NOT the same!

$$\lim_{h \to 0^+} \frac{(-3+h+3) - (-3+3)}{h} = \frac{+h}{h} \xrightarrow{h \to 0} 1$$

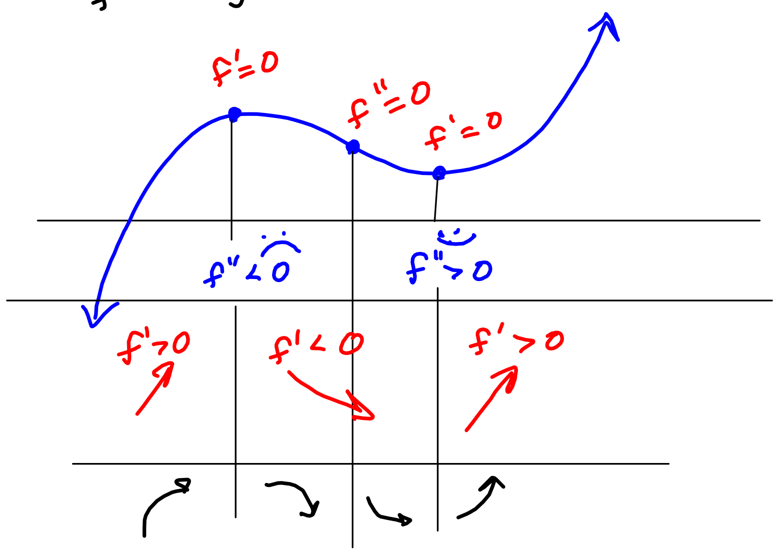
Higher Derivatives

f - Position

f' - speed/velocity

f'' - acceleration (concavity)

f''' - jerk



sketch the graph of $f(x) = x^3 - 6x^2 + 3x + 10$

$= (x+1)(x-2)(x-5)$

$3x^2 - 12x + 3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^3 - 6(x+h)^2 + 3(x+h) + 10 - \left[x^3 - 6x^2 + 3x + 10\right]}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 6(x^2 + 2xh + h^2) + 3x + 3h + 10 - x^3 + 6x^2 - 3x - 10}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 - \cancel{6x^2} \overset{+0?}{} - 12xh - 6x^2 + 3x + 3h + 6x^2 - 3x - 10}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 - 12xh + 3h}{h} = \frac{h(3x^2 + 3xh + h^2 - 12x + 3)}{h}$$

$$\xrightarrow{h \to 0} \boxed{3x^2 - 12x + 3 = f'(x)} \overset{\text{SET}}{=} 0$$

$\Rightarrow x^2 - 4x + 1 = 0$

$x^2 - 4x + 2^2 = -1 + 4$

$(x-2)^2 = 3$ $\qquad x = 2 \pm \sqrt{3}$

$$f''(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 12(x+h) + 3 - 3x^2 + 12x - 3}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2) - 12x - 12h + 3 - 3x^2 + 12x - 3}{h}$$

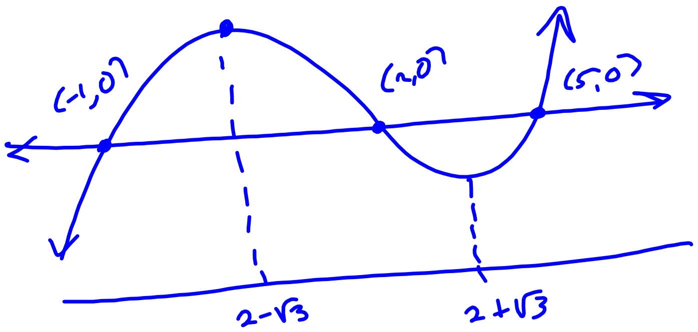
$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{- 12x} - 12h \cancel{+3} \cancel{-3x^2} \cancel{+12x} \cancel{-3}}{h}$$

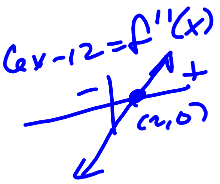
$$= \lim_{h\to 0} \frac{6xh + 3h^2 - 12h}{h} = \lim_{h\to 0} (6x + 3h - 12) = \boxed{6x - 12 = f''}$$

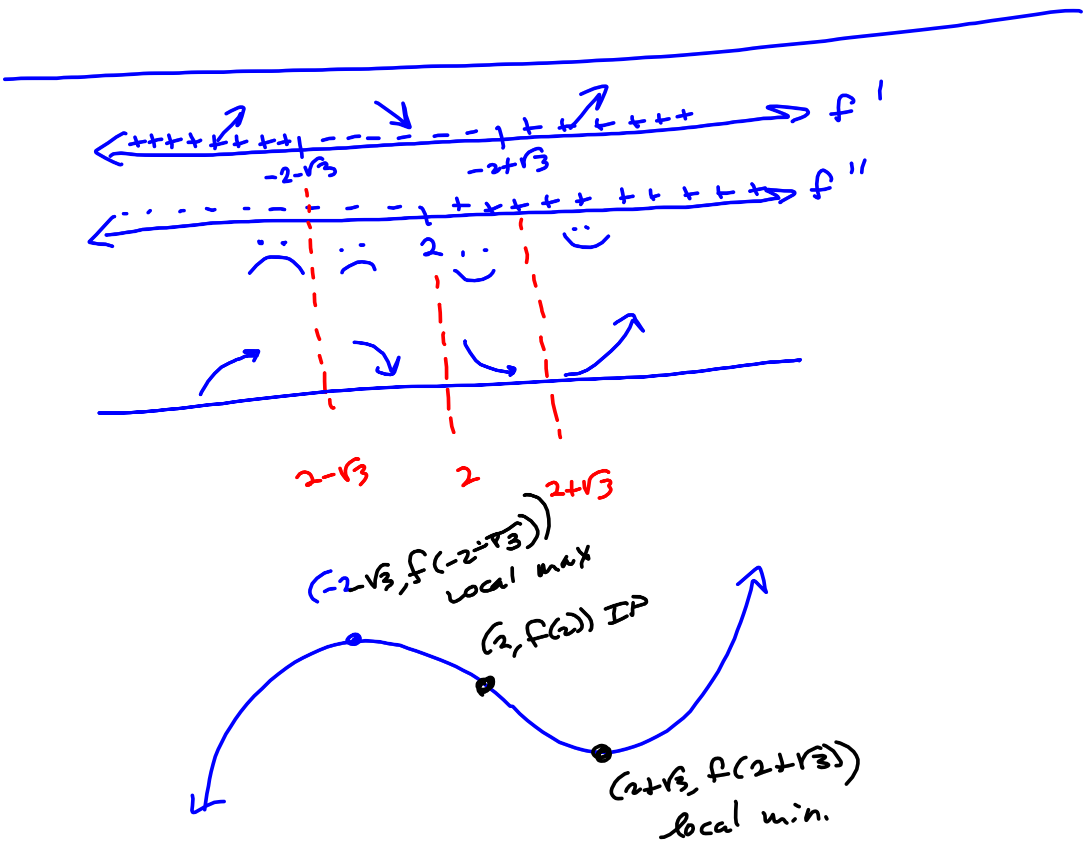
$$f''(x) = 0 \Rightarrow 6x - 12 = 0$$

$$6x = 12$$

$$\boxed{x = 2} \text{ I.P.}$$






f '
-2-√3
-2+√3
f ''
2
2-√3
2
2+√3
(-2-√3, f(-2-√3))
local max
(2, f(2)) IP
(2+√3, f(2+√3))
local min.

$f'(x)$ = derivative of f with respect to x

$= \frac{dy}{dx}$ = Liebniz Notation $= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$f'(7) = \left.\frac{dy}{dx}\right|_{x=7}$

$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$

$\frac{d}{dx}\left[c f(x)\right] = c \frac{df}{dx}$

$f(x) = x^3 - 5x^2 + 4x - 7$

$\Rightarrow 3x^2 - 10x^1 + 4x^0 - 0$

$= 3x^2 - 10x + 4$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

$f(x) = x^{\frac{2}{3}} \Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$