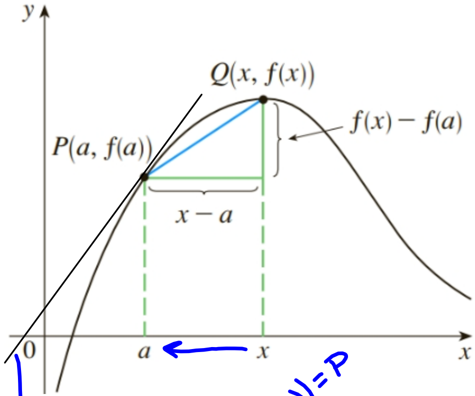
Section 2.1 - Derivatives and Rates of Change



tangent line to f(x) @ (a,f(a)) = P

We find the slope of the tangent line to *f* at *P*, which can be thought of as the slope of the curve at the point *P*.

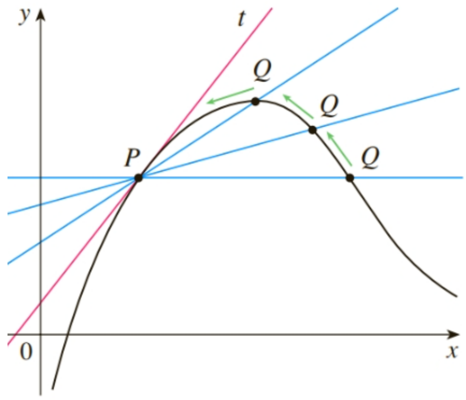
1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m_{tan} = m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \frac{y_2 - y_1}{x_2 - x}$$

provided that this limit exists.

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} \xrightarrow{x \to a} m_{tan}$$

$$\frac{f(x) - f(a)}{x - a} = \text{Difference Quotient}$$



Find the slope of the tangent line to

$f(x) = 4/x$ at $a = 1$.

want $\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{4}{x} - \frac{4}{1}}{x - 1}$

$$\lim_{x \to 1} \left(\frac{\frac{4 - 4x}{x}}{x - 1} \right) = \lim_{x \to 1} \frac{4 - 4x}{x(x-1)} = \lim_{x \to 1} \frac{4(1-x)}{x(x-1)}$$

$$= \lim_{x \to 1} \frac{-4(x-1)}{x(x-1)} = \lim_{x \to 1} \frac{-4}{x} = \frac{-4}{1} = \boxed{-4 = m}$$

Point-slope form for a line through the point (x_1, y_1) with slope m:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y = m(x - x_1) + y_1}$$

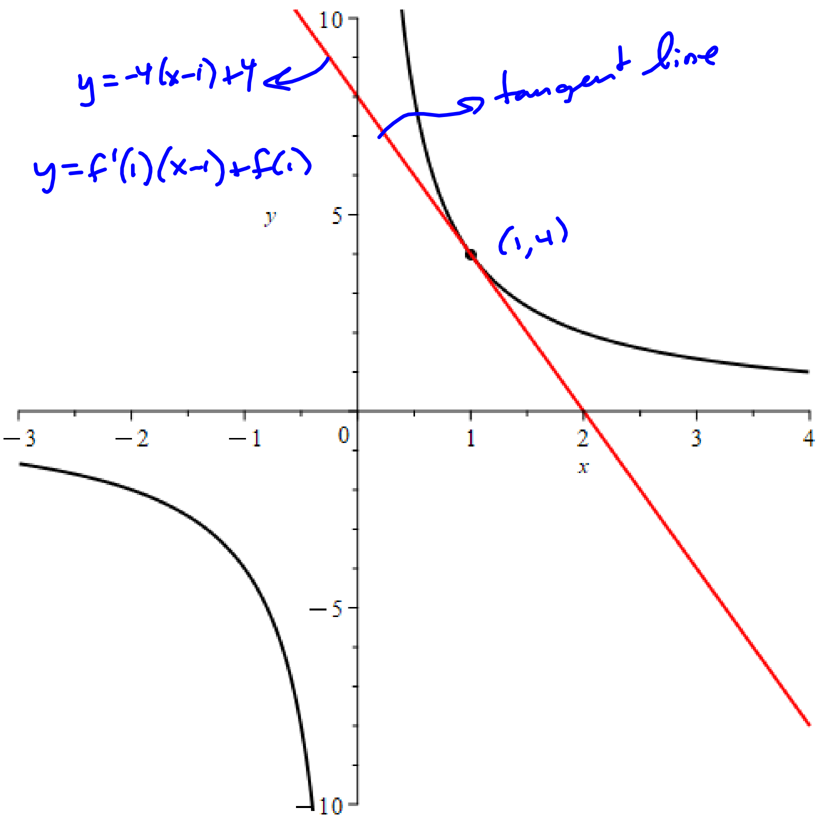
$$= y_1 + m(x - x_1)$$

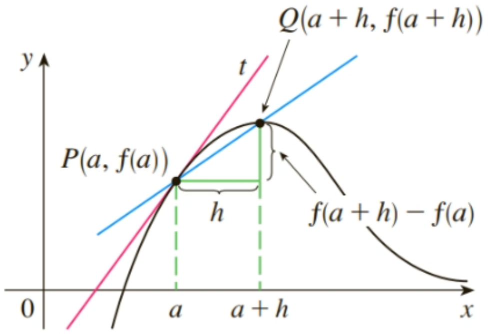
Eq'n of tan line:

$$y = m_{tan}(x - a) + f(a)$$

$$= f'(a)(x - a) + f(a)$$

where $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$


$y = -4(x-1)+4$
$y = f'(1)(x-1)+f(1)$
tangent line
(1,4)
10
5
y
0
−5
−10
−3
−2
−1
1
2
3
4
x



Reformulate the slope of the tangent to f @ x=a

x−a

Recall Q(x, f(x))

Think of x as being "h" units away from a.

Then $x = a+h$ &

$x - a = h$

Then $\frac{f(x)-f(a)}{x-a} =$

$$= \frac{f(a+h)-f(a)}{h}$$

$$\Rightarrow m_{tan} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Repeat the previous limit:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{4}{a+h}-\frac{4}{a}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{4}{a+h}-\frac{4}{a}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{4a-4(a+h)}{a(a+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{4a-4a-4h}{a(a+h)}\right] = \lim_{h\to 0}\frac{1}{\cancel{h}}\left[\frac{-4\cancel{h}}{a(a+h)}\right] = \lim_{h\to 0}\frac{-4}{a(a+h)} = \frac{-4}{a(a)} = \frac{-4}{a^2}$$

$$f'(a) = -\frac{4}{a^2}$$

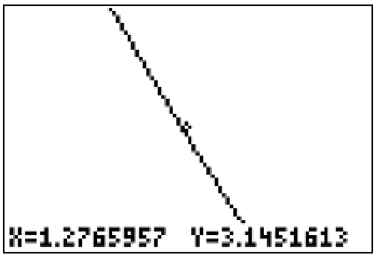
I didn't meant to leave the *a* as a letter, but it's actually better that way, and the book takes too stinking long to come out and say it.

So, $a = 1 \Rightarrow$ we have $f'(a) = -\frac{4}{1^2} = -4 = m_{tan}$

& tangent line is

$$y = f'(a)(x-a) + f(a)$$

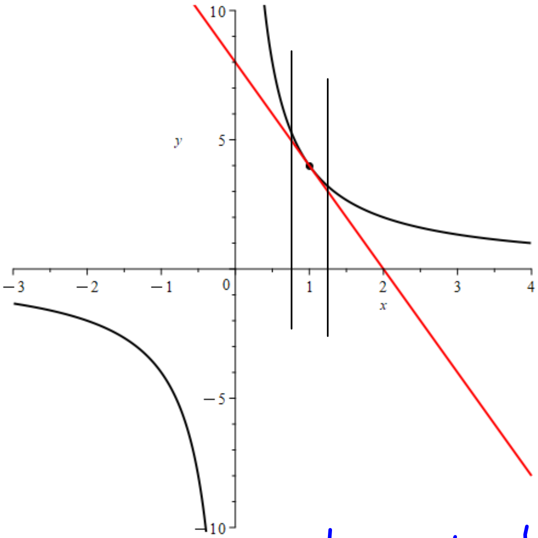
$$= -4(x-1) + \frac{4}{1} = \boxed{-4(x-1) + 4 = y}$$



When we zoom WAY IN on that point, we see that the curve looks pretty straight. We're basically looking at a very short segment of a curve.

"Smooth Curves Are Locally Linear."

The tangent line at *a* is pretty close to the function in the vicinity of $x = a$



If I stay close to $x = 1$, the function's pretty close to the tangent line. We often will use the "tangent line approximation" because it's more efficient than *f*, itself.

$f'(a)$ is the DERIVATIVE of $f(x)$ at $x = a$.

The book wanted us to do $f'(1)$ directly:

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{4}{1+h} - \frac{4}{1}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{4-4(1+h)}{1+h}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{4-4-4h}{1+h}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{-4h}{1+h}\right] = \lim_{h\to 0} \frac{-4}{1+h} = \boxed{-4 = f'(1)}$$

Find the eq'n of the tangent line to

$f(x) = x^2 - 2x$ @ $x = 1$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 2(x+h) - [x^2 - 2x]}{h}$$

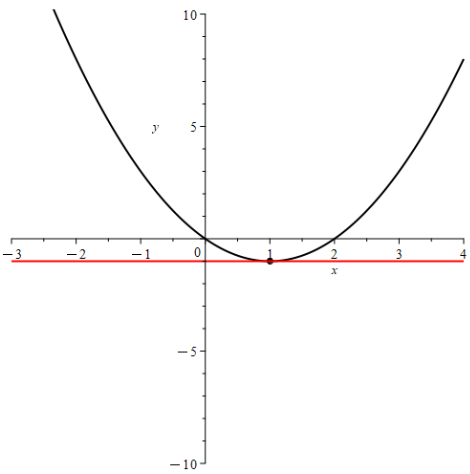
$$= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} = \frac{2xh + h^2 - 2h}{h} = \frac{h(2x + h - 2)}{h}$$

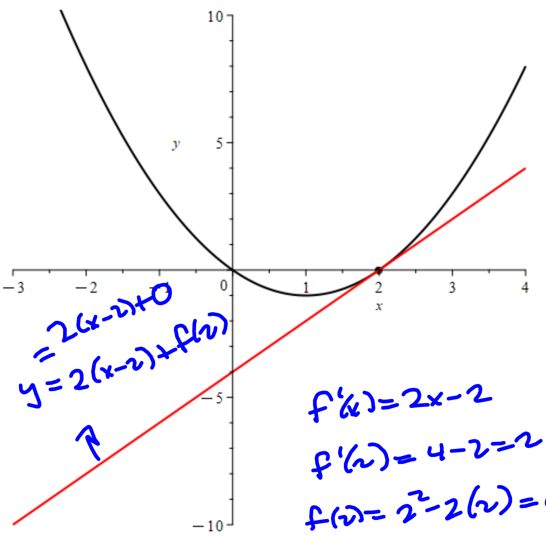
$$= 2x + h - 2 \xrightarrow{h \to 0} 2x - 2 = f'(x) \Rightarrow f'(1) = 2(1) - 2 = 0$$

$(h \neq 0)$

$y = f'(1)(x-1) + f(1) = f(1) = 1^2 - 2(1) = -1 = f(1)$

So $y = -1$ is tangent line equation!





$y = 2(x-2) + f(2)$

$= 2(x-2) + 0$

$f'(x) = 2x - 2$

$f'(2) = 4 - 2 = 2$

$f(2) = 2^2 - 2(2) = 0$

Average velocity is $\dfrac{\text{change in location}}{\text{change in time}} = \dfrac{\Delta y}{\Delta t}$ (Secant Line)

Instantaneous Velocity $= \lim\limits_{\text{change in time} \to 0} \dfrac{\text{change in location}}{\text{change in time}}$

Find an equation of the tangent line to $y = \text{sqrt}(x)$ at $x = 2$.

$f(x) = \sqrt{x}$, $a = 2$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \left(\frac{\overbrace{\sqrt{x+h} - \sqrt{x}}^{(a-b)}}{h}\right)\left(\frac{\overbrace{\sqrt{x+h} + \sqrt{x}}^{(a+b)}}{\sqrt{x+h} + \sqrt{x}}\right) = a^2 - b^2$$

$$= \frac{(\sqrt{x+h})^2 - \sqrt{x}^2}{h(\sqrt{x+h} + \sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \xrightarrow{h \to 0} \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = f'(x)$$

$h \neq 0$ proviso

$$\Longrightarrow f'(a) = f'(2) = \frac{1}{2\sqrt{2}} = m_{tan}$$

$$\Longrightarrow y = f'(a)(x-a) + f(a)$$

$$= \frac{1}{2\sqrt{2}}(x-2) + f(2) = \frac{1}{2\sqrt{2}}(x-2) + \sqrt{2} = y = L(x)$$

$$\frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$f(2) = \sqrt{2}$ "L" for "linearization."