Find two functions whose limits don't exist but their sum does.

$\lim_{x\to 0} \frac{1}{x}$ & $\lim_{x\to 0} -\frac{1}{x}$:

limits don't exist, but $\lim_{x\to 0}\left(\frac{1}{x}+\left(-\frac{1}{x}\right)\right) = \lim_{x\to 0}(0) = 0.$

31. [–/2 Points] DETAILS SCALC9 1.6.067.

Is there a number a such that the following limit exists? (If an answer does not exist, enter DNE.)

$$\lim_{x\to -2} \frac{2x^2 + ax + a + 6}{x^2 + x - 2}$$

Find the value a.

$\frac{\text{stuff}}{x^2+x-2} = \frac{\sim}{(x+2)(x-1)}$

Want $x+2$ to be a factor of

$2x^2+ax+a+6$:

$x(2x+a) = x(2x+4)$

Ans: $a=14$?!

$2x^2+ax+a+6 = (x+2)(\text{something})$

$2x^2+4x-4\cdot x+ax+a+6$

$= 2x^2+4x-4x+ax+a+3x+6-3x$

$= 2x(x+2)+3(x+2)+ax+a-7x$

$=$ $ax+a-7x =$

$2x^2+ax+a+6 \xrightarrow{x\to -2} 0$ must be the case

Jesse saw this

$2(-2)^2+(-2)a+a+6=$

$= 8-2a+a+6=0 \Rightarrow$

$14-a=0 \Rightarrow$

$\boxed{14=a}$!

$2x^2+14x+14+6$

$= 2x^2+14x+20$

$= 2(x^2+7x+10)$

$= 2(x+5)(x+2)$

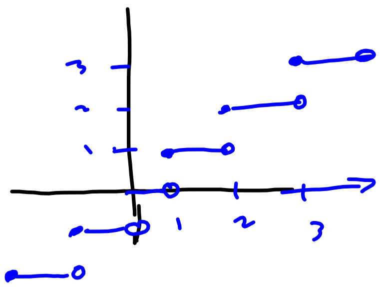
So $a=14 \Rightarrow$

$\frac{2x^2+ax+a+2}{x+2}$

$= \frac{2x^2+14x+20}{x+2}$

$= \frac{2(x+5)\cancel{(x+2)}}{\cancel{x+2}} = \underset{x\neq -2}{2(x+5)} \xrightarrow{x\to -2} \boxed{6}$

$[\![x]\!]$ = greatest integer $\le x$.



§1.6 #30 on WebAssign.

Find two functions whose limits don't exist but their sum does.

$\lim_{x\to 0} \frac{1}{x}$ & $\lim_{x\to 0} -\frac{1}{x}$:

limits don't exist, but $\lim_{x\to 0}\left(\frac{1}{x}+\left(-\frac{1}{x}\right)\right) = \lim_{x\to 0}(0) = 0.$

This is a cautionary tale about the precise wording of the arithmetic properties of limits.

The Big Theorem in §1.6 says:

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for $L, M \in \mathbb{R}$.

... then $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

You can't conclude that the sum of the limits exists, just because the limit of the sum exists.

$\lim(f+g)\ \exists \not\Rightarrow \lim f$ or $\lim g$ exist.

Section 1.7 - The Precise Definition of Limit

The limit as x approaches c of $f(x)$ is L means that I can keep $f(x)$ close to L by making x sufficiently close to c:

$\lim_{x\to c} f(x) = L$ means given any $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $|x-c| < \delta$, $|f(x)-L| < \varepsilon$

How close x is to c, How close f is to L

Our goal!

ε - Epsilon

δ - Delta

Keep $|f(x)-L| < \varepsilon$

Want $|f(x)-L| < \varepsilon$

Need $-\varepsilon < f(x) - L < \varepsilon$

i.e. $L-\varepsilon < f(x) < L+\varepsilon$

Need to keep $f(x)$ inside the ε-tube about $y=L$.

Claim! $\lim_{x\to 2} (3x-2) = 4$

Proof: Let $\varepsilon > 0$. Then, where the magic happens

$|f(x)-L|$

$= |3x-2-4| = |3x-6|$

$= 3|x-2|$

wait! that's how close to 2 the x is! we're using δ as the upper bound for that

$= 3|x-2| < 3\delta$

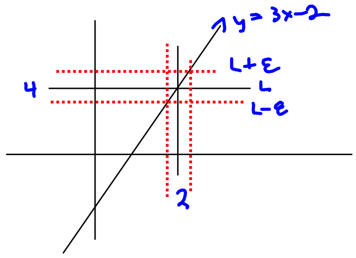
Now remember, we're trying to make $|f(x)-L| < \varepsilon$ so if we make $3\delta = \varepsilon$, we'll have it!

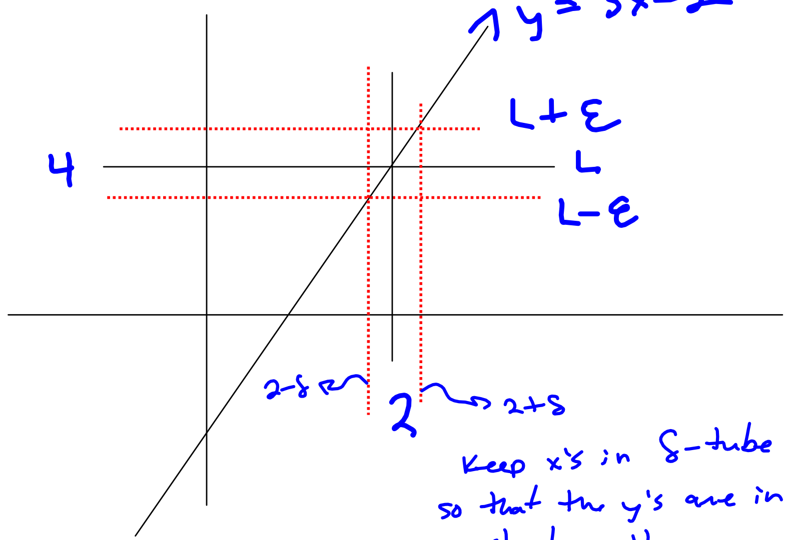
This means $\delta = \frac{\varepsilon}{3}$ works!

Define $\delta = \frac{\varepsilon}{3}$. Then, if $|x-2| < \delta$, we have $|f(x)-L| = |3x-2-4|$

$= |3x-6| = 3|x-2| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$

and we are done! We PROVED it!





Keep x's in δ-tube about $x=2$, so that the y's are in the ε-tube about $y=4$

2 **Precise Definition of a Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

No touching! $x = a$ is forbidden.
x as CLOSE to a as you wish.

Claim: $\lim_{x \to 3} (2x - 5) = 1$

Proof: Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{2}$.
Then, if $0 < |x - 3| < \delta$, we have
$|f(x) - L| = |2x - 5 - 1| = |2x - 6| = 2|x - 3| < 2\delta = 2\left(\frac{\varepsilon}{2}\right)$
$= \varepsilon$ ■

Next time, I'll show you how to handle a quadratic limit (and hence, any polynomial limit).

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of ± 5 cm^2 in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

$1000 - 5 = 995$
$1000 + 5 = 1005$

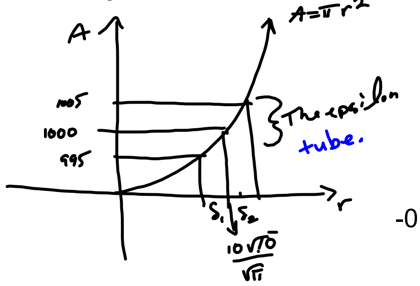
$\varepsilon = 5\text{ cm}^2$
want area A between 995 & 1005

(a) $\pi r^2 = 1000\text{ cm}^2$

$r^2 = \frac{1000}{\pi}$ $\quad \sqrt{r^2} = \sqrt{\frac{1000}{\pi}}$ $\quad |r| = \sqrt{\frac{1000}{\pi}}$

$r = \pm\sqrt{\frac{1000}{\pi}} \longrightarrow r = \frac{10\sqrt{10}}{\sqrt{\pi}} = \frac{10\sqrt{10}}{\sqrt{\pi}} \approx$

We want that number, $\pm 5\text{ cm}^2$



$\delta_2 = \frac{\sqrt{1005} - 10\sqrt{10}}{\sqrt{\pi}} \approx$ 0.04454749

$\delta_1 = \frac{10\sqrt{10} - \sqrt{995}}{\sqrt{\pi}} \approx .04465900$

-0.04465900

$\frac{\sqrt{1005}}{\sqrt{\pi}}$ is closer to $\frac{10\sqrt{10}}{\sqrt{\pi}}$ than $\frac{\sqrt{995}}{\sqrt{\pi}}$ is, so, we use that as our tolerance in the x-direction

Use $\delta_2 = \frac{\sqrt{1005} - 10\sqrt{10}}{\sqrt{\pi}} \approx$ 0.04454749

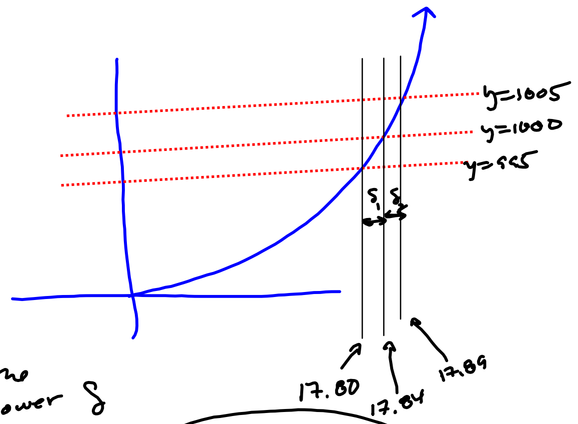
as it's the smaller of the 2.
If we're within .04454749, we're close enough on both sides of $\frac{10\sqrt{10}}{\sqrt{\pi}}$

§1.7: ε = hoped-for area tolerance (output variable)
δ = tolerance we set on the input variable

$\frac{\sqrt{1005}}{\sqrt{\pi}} \approx$ 17.88578865

$\frac{10\sqrt{10}}{\sqrt{\pi}} \approx$ 17.84124116

$\frac{\sqrt{995}}{\sqrt{\pi}} \approx$ 17.79658216



Find the narrower δ

$\delta_2 \approx$ 0.04454749

$\delta_1 \approx$ -0.04465900

Smaller one works for both! δ_2 is my δ for guaranteeing $995 < \text{Area} < 1005$

How close to a particular *x* do we have to be in order to keep *f*(*x*) sufficiently close to a particular *y*-value?

Want δ such that if $|x-a|<\delta$, we have $|f(x)-L|<\varepsilon$.

Claim:
$$\lim_{x\to 3}(2x+7)=13.$$

Proof:
Let $\varepsilon>0$ be given.

Scratch: want $|2x+7-13|<\varepsilon$

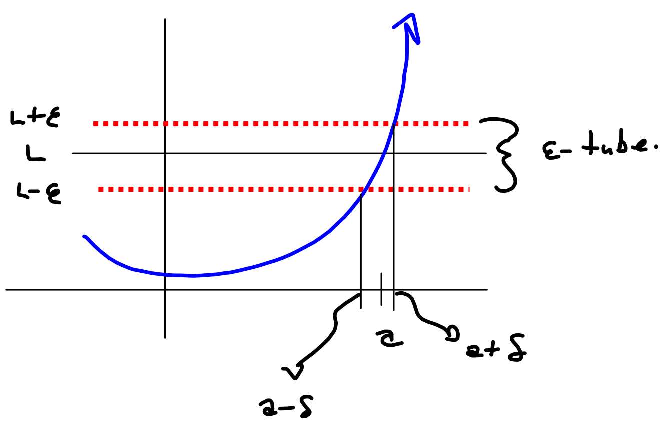
$\iff |2x-6|<\varepsilon$

$\iff 2|x-3|<\varepsilon$

$|x-3|<\frac{\varepsilon}{2}$

Define $\delta=\frac{\varepsilon}{2}$. Then $0<|x-3|<\frac{\varepsilon}{2} \implies$

$|2x+7-13| = |2x-6| = 2|x-3| < 2\cdot\frac{\varepsilon}{2} = \varepsilon$ ∎



That's the extent of what you'll be tested on under a time control (except for bonus).

Claim: $\lim_{x \to 5} (3x-2) = 13$

Proof

Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{3}$ so that if $0 < |x-5| < \delta$, we have $|3x-2-13| = |3x-15| = 3|x-5| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ ∎

Save this one for Wednesday. Quadratic Proof.

Claim: $\lim_{x \to 2} (x^2 - 5x + 7) = 1$

Scratch 1^{st}, assume $\delta \le 1$, i.e., $|x-2| \le 1$.

Want $|x^2 - 5x + 7 - 1| = |x^2 - 5x + 6|$

$= \underbrace{|x-3|}_{<\,?}\underbrace{|x-2|}_{<\,\delta} < \varepsilon$

Now $\delta \le 1$ means $1 \le x \le 3$

$\Rightarrow 1-3 \le x-3 \le 0$, i.e.

$-2 \le x-3 \le 0$

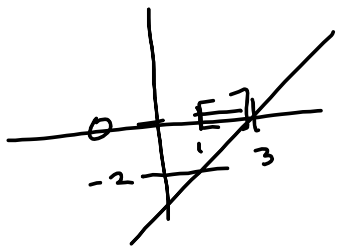
Do on Wednesday.

How bad can $|x-3|$ get on $[1, 3]$?

No worse than 2.

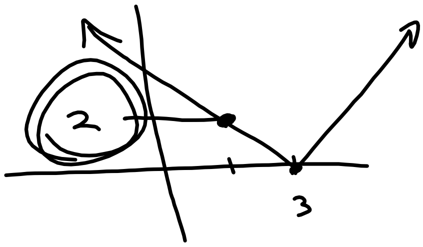
$-2 \le x-3$ means

$-(-2) \ge -(x-3)$



what's the biggest this gets in absolute value?

$|x-3|$



So $|x-3| \le 2$ if $1 \le 2 \le 3$

Proof

Let $\epsilon > 0$ be given. Define $\delta = \min\{1, \frac{\epsilon}{2}\}$.

Then $0 < |x-2| < \epsilon \Longrightarrow |x^2-5x+7-1|$

$= |x^2-5x+6| = |x-3||x-2| \le 2|x-2| < 2 \cdot \frac{\epsilon}{2} = \epsilon$ $\blacksquare$

§1.8 Continuity

$f(x)$ is continuous at $x=c$ for $c \in$ domain of f.

$$\lim_{x \to c} f(x) = f(c)$$

No holes. No breaks @ $x=c$.

$f(x)$ is continuous on an Interval I

if $f(x)$ is continuous at c $\underbrace{\text{for all}}_{\forall}$ $c \in I$.

cont$\underline{s}$

99% of Continuity Questions are actually just Domain Questions.

Almost any function we can write is continuous on its domain.

Domain: Just 2 things, really....

$\dfrac{\text{STUFF}}{0}$ BAD

$\sqrt[2n]{\text{negative}}$ BAD

$f(x) = \dfrac{\sin(x)}{x+1}$

is continuous on its Domain

Need $x+1 \neq 0$

$\Rightarrow x \neq -1 \Rightarrow$

f is cont$\underline{s}$ on $(-\infty,-1)\cup(-1,\infty)$

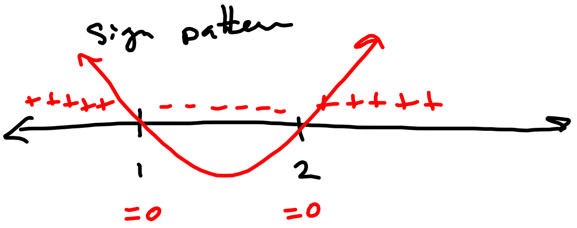
$= \mathbb{R} \setminus \{-1\}$

$f(x) = \sqrt{x^2-3x+2}$

$x^2-3x+2 \geq 0$

$(x-1)(x-2) \geq 0$

Sign pattern



want ≥ 0

Yes | No | Yes

1 Yes, 2 Yes

$\Rightarrow \mathcal{D} = (-\infty, 1] \cup [2, \infty)$

$\infty = \infty + 1 \rightarrow$

$-\infty = -\infty$

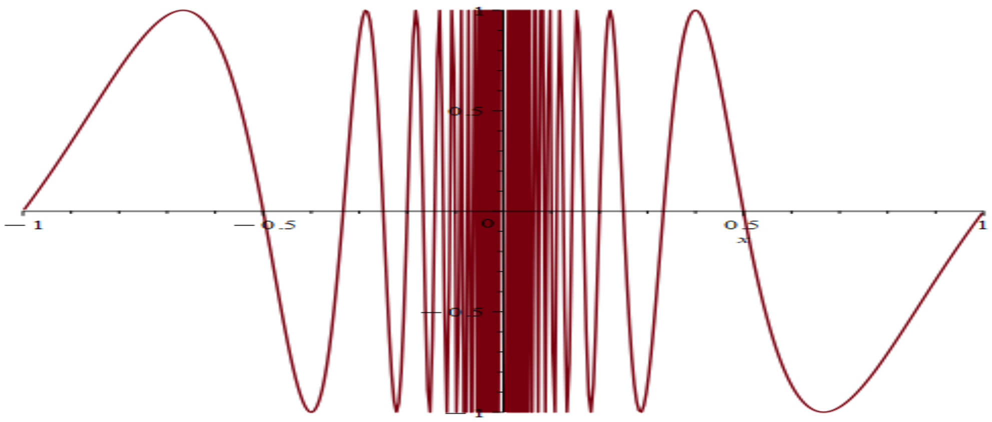
$0 = 1$?!

No way

∞ ain't real.

Recall $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist

$\nexists$



So it's not cnt^s @ $x=0$, But

$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous for all real x

"is cnt^s $\forall x \in \mathbb{R}$"

$\lim_{x\to 0} f(x) = 0$

$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$

$-x \leq x\sin\left(\frac{\pi}{x}\right) \leq x$ if $x > 0$.

& $\lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$ $\Longrightarrow$

$\lim_{x\to 0} x\sin\left(\frac{\pi}{x}\right) = 0$ by Squeeze Theorem.

One-Page, 2-sided cheat sheet permitted on all tests.