§1.6 Limit Laws

**Mainly, we would LIKE to just be able to plug in the limiting value, and most of the time we can, but of course, the limits for calculus always involve situations where you can't.**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

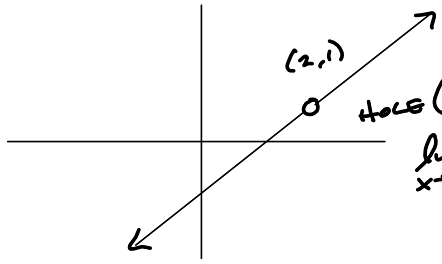
$\lim_{x \to c} f(x) = f(c)$ most of the time.

Polynomials, Rational Functions (on their domain), sine, cosine, tangent, ...

$\uparrow$ on its domain

$$\lim_{x \to \frac{\pi}{6}} \sin(x) = \sin\left(\frac{\pi}{6}\right)$$

$$\lim_{x \to 2} \frac{x^2-3x+2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x-2} = \lim_{x \to 2} (x-1) = 1.$$




Graph of $\frac{x^2-3x+2}{x-2} = f(x)$

Hole @ $x=2$, but

$\lim_{x \to} f(x) = 1.$

**If two** $\overbrace{\text{functions}}^{f \,\&\, g}$ **agree everywhere except possible** $x = c$**, then**

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Fact: for small $\theta$,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta\to 0} \cos\theta = 1$$

$$\lim_{\theta\to 0} 1 = 1$$

$\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. This is key to showing $\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos(x)$

§2.4

Squeeze Theorem

if $f(x) \le g(x) \le h(x)$ for $x$ near $c$.

and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then

$\lim_{x\to c} g(x) = L$.

If $\lim f$ & $\lim g$ exist, then.

Just what we'd hope & expect.

$$\lim (f \pm g) = \lim f \pm \lim g$$

$$\lim (fg) = (\lim f)(\lim g)$$

$$\lim \left(\frac{f}{g}\right) = \frac{\lim f}{\lim g} \quad (\text{provided } \lim g \neq 0)$$

$$\lim_{x\to 2} \frac{(x-5)(x+2)}{x+3} = \frac{\lim_{x\to 2}(x-5)\lim_{x\to 2}(x+2)}{\lim_{x\to 2}(x+3)}$$

$$= \frac{\left(\lim_{x\to 2}(x) - \lim_{x\to 2}(5)\right)\left(\lim_{x\to 2}(x) + \lim_{x\to 2}(2)\right)}{\lim_{x\to 2}(x) + \lim_{x\to 2}(3)} \quad \text{UGH!}$$

These five laws can be stated verbally as follows:

**Sum Law** 1. The limit of a sum is the sum of the limits.

**Difference Law** 2. The limit of a difference is the difference of the limits.

**Constant Multiple Law** 3. The limit of a constant times a function is the constant times the limit of the function.

**Product Law** 4. The limit of a product is the product of the limits.

**Quotient Law** 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

**Power Law** **6.** $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where $n$ is a positive integer

**We already know this by Product Law! Just keep applying it over and over, *n* times!**

**7.** $\lim_{x \to a} c = c$ **8.** $\lim_{x \to a} x = a$

**#s 6 - 8 are what tell us that** $\lim_{x \to 1} (x^2+5x+7) = 1^2+5(1)+7 = 13$

$$\lim_{x \to 1} \frac{x^2-5x-7}{x+3} = \frac{1^2-5(1)-7}{1+3} = \frac{-11}{4}$$

$$\lim_{t \to -2} \frac{t^4 - 4}{2t^2 - 3t + 4} = \frac{16-4}{18} = \frac{12}{18} = \frac{2}{3}$$

$2(-2)^2 - 3(-2) + 4 = 8 + 6 + 4 = 18$

**This was one that didn't require any histrionics, because *x* = -2 is in the domain of the rational function!**

$$\lim_{t \to -2} \frac{t^4 - 4}{2t^2 - 3t + 4} = \frac{\lim_{t \to -2} (t^4 - 4)}{\lim_{t \to -2} (2t^2 - 3t + 4)} \qquad \text{[Limit Law 5]}$$

$$= \frac{\lim_{t \to -2} t^4 - \lim_{t \to -2} 4}{2 \lim_{t \to -2} t^2 - 3 \lim_{t \to -2} t + \lim_{t \to -2} 4} \qquad \text{[1, 2 and 3]}$$

$$= \frac{16 - 4}{2(4) - 3(-2) + 4} \qquad \text{[9, 7, and 8]}$$

$$= \frac{2}{3}$$

In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

$\lim_{v \to c} L(v) = 0$

expresses the length $L$ of an object as a function of its velocity $v$ with respect to an observer, where $L_0$ is the length of the object at rest and $c$ is the speed of light.

Find $\lim_{v \to c^-} L$.

If $\lim_{x \to 1} \frac{f(x) - 7}{x - 1} = 5$, find $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1} \frac{f(x)-7}{x-1} = 5 \implies \frac{\lim_{x \to 1} f(x) - 7}{\lim_{x \to 1} (x) - 1}$

I'd say $\lim_{x \to 1} f(x) = 7$,

or else $\lim_{x \to 1} \frac{f(x)-7}{x-1} \nexists$ !

The $x-1 \to 0$

so $f(x) - 7 \to 0$, too !

Is there a number $a$ such that the following limit exists? (If an answer does not exist, enter DNE.)

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 9}{x^2 + x - 2} = \lim_{x\to -2} f(x)$$

Find the value $a$.

$x^2+x-2 = (x+2)(x-1) \xrightarrow{x\to -2} 0$, so the numerator must ALSO approach 0 as $x \to -2$

$3x^2+ax+a+9 \xrightarrow{x\to -2} 0$

So it must contain a factor of $x+2$

$3(-2)^2-2a+a+9 = 12-a+9 = 21-a=0 \longrightarrow \boxed{21=a}$

Check:

PRO $\dfrac{3x^2+21x+21+9}{x^2+x-2} = \dfrac{3x^2+21x+30}{x^2+x-2}$

$= \dfrac{3(x^2+7x+10)}{(x+2)(x-1)} = \dfrac{3(x+2)(x+5)}{(x+2)(x-1)} = \boxed{\dfrac{3(x+5)}{x-1} \; (x \neq -2)} \xrightarrow{x\to -2} \dfrac{3(-2+5)}{-2-1}$

$= \dfrac{9}{-3} = -3.$

Book doesn't do.

Book $\lim\limits_{x\to -2} \dfrac{3x^2+21x+30}{x^2+x-2} = \lim\limits_{x\to -2} \dfrac{3\cancel{(x+2)}(x+5)}{\cancel{(x+2)}(x-1)} = \lim\limits_{x\to -2} \dfrac{3(x+5)}{x-1} = \dfrac{3(3)}{-2-1} = \dfrac{9}{-3} = -3$

Don't need to say "$x \neq -2$"

**If the limit exists and the denominator is zero at that limit, then the numerator must also approach zero or you have a 7/0 or -5/0 situation, which doesn't exist.**