

Derivative of logarithmic functions:

$$y = \ln(x) \rightarrow e^y = x$$

$$\Rightarrow \frac{d}{dx}[e^y] = e^y y' = 1 \rightarrow$$

$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} !$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x} \quad \text{and} \quad \int \frac{1}{x} dx = \int \frac{dx}{x} = \ln(x) + C \quad (\text{so } x > 0 \text{ is needed})$$

See Example 7 in text.

Actually

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$u = f(x) \rightarrow$$

$$du = f'(x) dx$$

$$\ln|u| + C = \int \frac{du}{u} = \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

Recall

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \sec(x) dx = ?$$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int \csc(x) dx = ?$$

$$\int \sec^2(x) dx = \tan(x) + C$$

$$\int \tan(x) dx = ?$$

$$\int \csc^2(x) dx = -\cot(x) + C$$

$$\int \cot(x) dx = ?$$

$$\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx = - \int \frac{-\sin(x)}{\cos(x)} dx = - \int \frac{f'(x)}{f(x)} dx,$$

where  $f(x) = \cos(x) \rightarrow$  we have

$$\begin{aligned}
 -\ln |f(x)| + C &= -\ln |\cos(x)| + C = -1 \ln |\cos(x)| + C \\
 &= \ln(|\cos(x)|^{-1}) + C = \ln |\cos(x)^{-1}| + C \\
 &= \ln \left| \frac{1}{\cos(x)} \right| + C = \ln |\sec(x)| + C = \int \tan(x) dx
 \end{aligned}$$

$$\int \sec(x) dx = \int \sec(x) \left( \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} \right) dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)} dx = \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

$$\begin{aligned}
 \text{Now } f(x) &= \sec(x) + \tan(x) \\
 \Rightarrow f'(x) &= \sec(x)\tan(x) + \sec^2(x)
 \end{aligned}$$

$$\begin{aligned}
 &= \ln |\sec(x) + \tan(x)| + C \\
 &= \int \sec(x) dx
 \end{aligned}$$

Now, you can derive  $\int \csc(x) dx$

$$\& \int \cot(x) dx.$$

Ch 4: Exponential and Logarithmic Functions	365
Ch 4: Introduction	365
4.1: Exponential Functions	366
4.2: The Natural Exponential Function	374
4.3: Logarithmic Functions	380
4.4: Laws of Logarithms	390
4.5: Exponential and Logarithmic Equations	396
4.6: Modeling with Exponential Functions	406

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Logarithmic and Exponential Functions  
in College Algebra**

Applications of  $\frac{d}{dx} [\ln(f(x))] = \frac{f'(x)}{f(x)}$

$$\begin{aligned} & \frac{d}{dx} \left[ \ln \left( \frac{\sqrt[5]{x^2-5} (x^5+2x)^2}{\sin(x)} \right) \right] \\ &= \frac{d}{dx} \left[ \ln \left( \frac{(x^2-5)^{\frac{1}{5}} (x^5+2x)^2}{\sin(x)} \right) \right] \\ &= \frac{d}{dx} \left[ \ln((x^2-5)^{\frac{1}{5}}) + \ln((x^5+2x)^2) - \ln(\sin(x)) \right] \\ &= \frac{d}{dx} \left[ \frac{1}{5} \ln(x^2-5) + 2 \ln(x^5+2x) + \ln(\sin(x)) \right] \\ &= \frac{1}{5} \left( \frac{2x}{x^2-5} \right) + 2 \left( \frac{5x^4+2}{x^5+2x} \right) + \frac{\cos(x)}{\sin(x)} \\ & \quad \rightarrow \cot(x). \end{aligned}$$

Final bit of theory for the semester:

Logarithmic Differentiation. Very nice for  $\frac{d}{dx} [f(x)^{g(x)}]$

$$\frac{d}{dx} \left[ \sin(x)^{\cos(x)} \right] = \frac{d}{dx} [y], \text{ where}$$

$$y = \sin(x)^{\cos(x)} \longrightarrow$$

$$\ln(y) = \ln(\sin(x)^{\cos(x)}) = \cos(x) \ln(\sin(x))$$

$$\frac{d}{dx} [LHS = RHS] :$$

$$\frac{y'}{y} = -\sin(x) \ln(\sin(x)) + \cos(x) \cdot \frac{\cos(x)}{\sin(x)} \longrightarrow$$

$$y' = \left( \text{W HOLE MESS} \right) \cdot y$$

$$= \left( -\sin(x) \ln(\sin(x)) + \frac{\cos^2(x)}{\sin(x)} \right) (\sin(x)^{\cos(x)}) = y'$$

$$\frac{d}{dx} \left[ \frac{(x^2+5x)^{13} (x^3+11x^2)^7}{(x^5+7)^3 (\sin^2(x))} \right] = \frac{d}{dx} [y] \Rightarrow$$

$$\ln(y) = 13 \ln(x^2+5x) + 7 \ln(x^3+11x^2) - 3 \ln(x^5+7) - 2 \ln(\sin^2(x))$$

$$\Rightarrow y' = \left( 13 \left( \frac{2x+5}{x^2+5x} \right) + 7 \left( \frac{3x^2+22x}{x^3+11x^2} \right) - 3 \left( \frac{5x^4}{x^5+7} \right) - 2 \cot(x) \right) \left( \frac{(x^2+5x)^{13} (x^3+11x^2)^7}{(x^5+7)^3 (\sin^2(x))} \right)$$

Find  $y'$  if  $y = \ln(x^2 - \sin^2(y))$

$$\Rightarrow y' = \frac{2x - \cos(y)y'}{x^2 - \sin^2(y)} = \frac{2x}{x^2 - \sin^2(y)} - \frac{\cos(y)y'}{x^2 - \sin^2(y)}$$

$$y' + \frac{\cos(y)y'}{x^2 - \sin^2(y)} = \frac{2x}{x^2 - \sin^2(y)} \Rightarrow$$

$$y' \left( 1 + \frac{\cos(y)}{x^2 - \sin^2(y)} \right) = \frac{2x}{x^2 - \sin^2(y)}$$

$$y' = \frac{\frac{2x}{x^2 - \sin^2(y)}}{\frac{x^2 - \sin^2(y) + \cos(y)}{x^2 - \sin^2(y)}} = \frac{2x}{x^2 - \sin^2(y)} \cdot \frac{x^2 - \sin^2(y)}{x^2 - \sin^2(y) + \cos(y)}$$

$$= \frac{2x}{x^2 - \sin^2(y) + \cos(y)} = y'$$