Approximate Area under a curve, Riemann Sums, and the Definite Integral.

Sections 4.1 - 4.3-ish

1. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2 + \text{lower degree}}{2}$$

2. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + \text{lower degree}}{6}$$

3. $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4 + \text{lower degree}}{4}$$

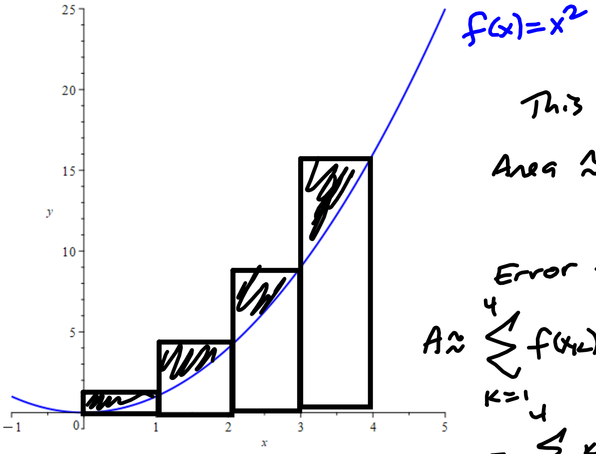
output = value, sum, plot, or animation

Approximate the area under the curve:

$f(x) = x^2$ on $[a,b] = [0,4]$, using $n = 4, 10, 100$?! rectangles.

Right Endpoints:

$n = 4$



This is right endpoints picture

Area $\approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 >$ Actual Area.

Error = shaded part.

$$A \approx \sum_{k=1}^{4} f(x_k)\Delta x = \sum_{k=1}^{4} f(k)\cdot 1$$

$$= \sum_{k=1}^{4} k^2 = 1^2+2^2+3^2+4^2$$

$$= \frac{4(4+1)(2(4)+1)}{6} = \frac{4(5)(9)}{6} =$$

$$= 2(5)(3) = 30 \approx \text{Area.}$$

$x_1 = 1$

$x_2 = 1 + \Delta x = 2$

$x_3 = 2 + \Delta x = 3$

$x_4 = 3 + \Delta x$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1 = \Delta x$$

8 rectangles (right endpoints)

$x_1 = a + \Delta x$

$x_2 = x_1 + \Delta x$

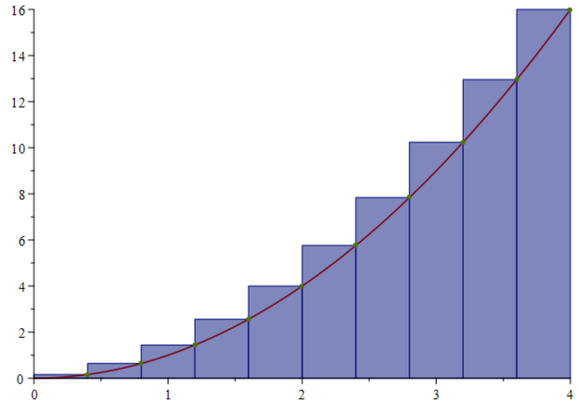
$x_3 = x_2 + \Delta x$

$\vdots$

$x_n = x_n + \Delta x = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) = a + b - a = b$

Approximate Area using Right endpoints:

$\boldsymbol{n = 10}$**:** 24.64000000



A right Riemann sum approximation of $\int_0^4 f(x)\,dx$, where $f(x) = x^2$ and the partition is uniform.

The approximate value of the integral is 24.64000000. Number of subintervals used: 10.

$a = x_0$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\vdots$

$x_k = a + k\Delta x$

$= 0 + k \cdot \frac{b-a}{n} = 0 + \frac{4}{n}k$

$\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$\sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n} \left(\frac{b-a}{n}k\right)^2 \cdot \frac{b-a}{n}$

$= \frac{4}{4} \sum_{k=1}^{4} \left(\frac{4}{4}k\right)^2 = 30$

$= 4$ is our 'n.'

Consider the following theorem.

4.2 #5 If f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

As $\Delta x \to 0$

$n \to \infty$ in

$\sum_{k=1}^{n} f(x_k)\Delta x$

Use the given theorem to evaluate the integral.

$$\int_3^8 (24 - 8x)\,dx$$

$a = 3,\ b = 8$

$\Delta x = \frac{b-a}{n} = \frac{8-3}{n} = \frac{5}{n}$

$x_k = a + k\Delta x = 3 + \frac{5}{n}k$

$f(x) = 24 - 8x$

$f(x_k) = \left[24 - 8\left(3 + \frac{5}{n}k\right)\right]$

$24 - 24 - \frac{40}{n}k = -\frac{40}{n}k$

$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{5}{n}\sum_{k=1}^{n}\left(24 - 8\left(3 + \frac{5}{n}k\right)\right)$

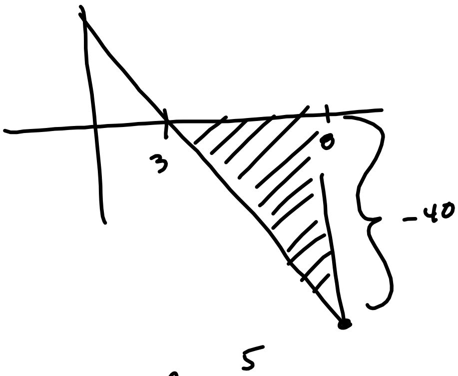
$= \frac{5}{n}\sum_{k=1}^{n} -\frac{40}{n}k = \frac{5}{n}\left(-\frac{40}{n}\right)\sum_{k=1}^{n} k = -\frac{5}{n}\left(\frac{40}{n}\right)\left[\frac{n(n+1)}{2}\right]$

$= -\frac{200}{n^2}\left[\frac{n^2}{2} + \frac{n}{2}\right] = -\frac{200n^2}{2n^2} + \frac{-200n}{2n^2} = -100 + \left(-\frac{100}{n}\right) \to 0$

$\xrightarrow{n \to \infty} -100$

$y = -8x + 24 = f(x) \Rightarrow f(8) = -64 + 24 = -40$

x	y
0	24
3	0



In the limit, we keep only the "big stuff":

$$\frac{n(n+1)}{2} = \frac{n^2+n}{2} = \frac{n^2}{2} + \frac{n}{2}$$

Goes Away

$$\frac{n(n+1)}{2} = \frac{n^2 + \text{lower degree}}{2}$$

$$= \frac{n^2 + \sim}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof (by Induction)

Note It works for $n=1$:

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1 \;!$$

Now, suppose it works for some $n \geq 1$.

WTS it works for $n+1$.

Assume $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for some $n \geq 1$.

Then $$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n+1 = \frac{n(n+1)}{2} + \frac{n+1}{1} \cdot \frac{2}{2}$$

$$= \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$$

$\Rightarrow$ It works for $n+1$!

Similar proofs for $\sum_{k=1}^{n} k^2$ & $\sum_{k=1}^{n} k^3$

4.2 #7

Consider the following.

$$\int_0^4 (x^2 - 3x)\,dx$$

(a) Find an approximation to the integral using a Riemann sum with right endpoints and $n = 8$.

$R_8 =$

(b) Draw a diagram to illustrate the approximation in part (a).

$a=0,\ b=4,\ \Delta x = \frac{b-a}{n} = \frac{4}{n} = \frac{4}{8} = \frac{1}{2}$

$x_k = a + k\Delta x = 0 + k\cdot\frac{4}{n} = \frac{4}{n}k$

$f(x_k) = x_k^2 - 3x_k = \left(\frac{4}{n}k\right)^2 - 3\left(\frac{4}{n}k\right)$

$= \frac{16k^2}{n^2} - \frac{12k}{n}$

$n=8:\ f(x_k) = \frac{16k^2}{64} - \frac{12k}{8} = \frac{k^2}{4} - \frac{3k}{2}$

$\frac{1}{2}\sum_{k=1}^{8}\left(\frac{k^2}{4} - \frac{3k}{2}\right) = \frac{1}{2}\left[\sum_{k=1}^{8}\frac{k^2}{4} - \sum_{k=1}^{8}\frac{3k}{2}\right]$

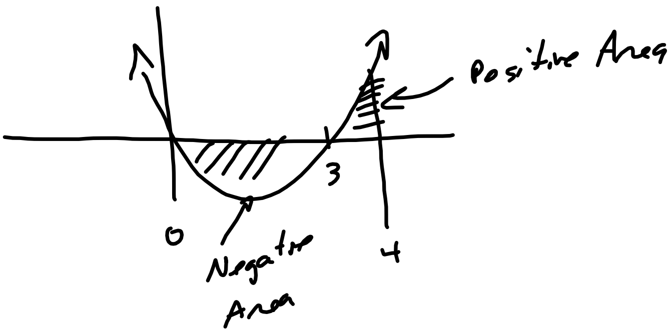
$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3 + \cdots}{3}$

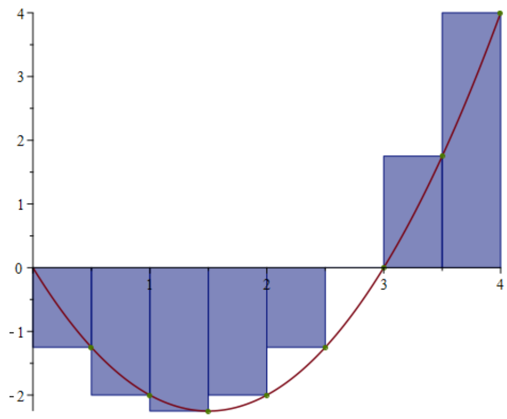
$= \frac{1}{2}\left(\frac{1}{4}\right)\sum_{k=1}^{8} k^2 - \frac{1}{2}\left(\frac{3}{2}\right)\sum_{k=1}^{8} k$

$$= \frac{1}{8}\left[\frac{8(8+1)(2(8)+1)}{6}\right] - \frac{3}{4}\left[\frac{8(8+1)}{2}\right]$$

$$= \frac{1}{\cancel{6\cdot 8}\,2}\left[\cancel{8}(\cancel{9}\,3)(17)\right] - \frac{3}{\cancel{4\cdot 2}}\left[\cancel{8}(9)\right]$$

$$= \frac{51}{2} - 27 = \frac{51}{2} - \frac{54}{2} = -\frac{3}{2}$$





A right Riemann sum approximation of $\int_0^4 f(x)\,dx$, where $f(x) = x^2 - 3x$ and the partition is uniform. The approximate value of the integral is -1.500000000. Number of subintervals used: 8.

Let's find the EXACT value of the integral:

$\int_0^4 (x^2 - 3x)\,dx$ $\qquad \Delta x = \frac{4}{n},$

$x_k = 0 + k\left(\frac{4}{n}\right) = \frac{4}{n}k$

$f(x_k) = \left(\frac{4}{n}k\right)^2 - 3\left(\frac{4}{n}k\right)$

$= \frac{16k^2}{n^2} - \frac{12k}{n}$

$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{4}{n} \sum_{k=1}^{n} \left(\frac{16k^2}{n^2} - \frac{12k}{n}\right)$

$= \frac{4 \cdot 16}{n^3} \sum_{k=1}^{n} k^2 - \frac{4(12)}{n^2} \sum_{k=1}^{n} k$

$= \frac{64}{n^3}\left[\frac{n^3 + \sim\sim}{3}\right] - \frac{48}{n^2}\left[\frac{n^2 + \sim\sim}{2}\right]$

$\xrightarrow{n \to \infty} \frac{64 n^3}{3n^3} - \frac{48}{n^2}\left[\frac{n^2}{2}\right] = \frac{64}{3} - \frac{48}{2} = \frac{64}{3} - \frac{24}{1} \cdot \frac{3}{3} = \frac{64 - 72}{3}$

$= \boxed{-\frac{8}{3} = \int_0^4 f(x)\,dx}$