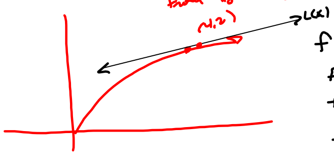
2.9 - Linear Approximations and Differentials

Recall the tangent line to *f* at (*xo*, *yo*).

$y = f'(x_0)(x-x_0) + y_0$

$= f'(x_0)\underbrace{(x-x_0)}_{\text{How far from } x_0 \text{ is } x?} + f(x_0)$



$f(x) = \sqrt{x}$.

Find the tangent line to $f(x)$ @ $(4,2)$

$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow$

$f'(x) = \frac{1}{2}(x^{-\frac{1}{2}}) = \boxed{\frac{1}{2\sqrt{x}} = f'(x)}$

$\Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

$y = \frac{1}{4}(x-4) + \sqrt{4}$

$L(x) = \frac{1}{4}(x-4) + 2 = y$

= Linearization of f(x) @ (4,2)

$= \frac{1}{4}x - 1 + 2 = \frac{1}{4}x + 1$

We can use this to approximate *f* in a neighborhood of *xo*

use the linearization @ (4,2) to approximate $\sqrt{4.1}$

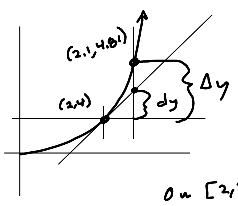
$L(4.1) = \frac{1}{4}(4.1-4) + 2$

$= \frac{1}{40} + 2\left(\frac{40}{40}\right) = \frac{1+80}{40} = \frac{81}{40}$

$evalf\left(\frac{81}{40}\right) = 2.025000000$

$evalf(\mathrm{sqrt}(4.1)) \approx 2.024845673$

$f(x) = x^2$ @ $(x,y) = (2,4)$



Find $L(2.1) = ?$

$f'(x) = 2x$

$f'(2) = 4$ $\quad dx = \Delta x$

$L(x) = 4\underbrace{(x-2)}_{dx=\Delta x} + 4$

$L(2.1) = 4(.1) + 4 = 4.4 \approx 2^2$

On $[2, 2.1]$, The net change is

$\Delta y = f(2.1) - f(2) = 4.1^2 - 4^2 = .81$

The linearization approximated this

$4.1^2 \approx 4.4$

$\Delta y \approx .4$ using linearization

$.4 = L(2.1) - L(2)$

$y = x^2$

$\frac{dy}{dx} = 2x$

$dy = 2x\,dx = f'(x)\,dx$ = Differential of y

Define $dx = \Delta x = x - x_0 = 2.1 - 2 = .1$

$\Delta y \approx dy = 2(2)(.1) =$

How much paint to put a .2 cm coat on a sphere of radius 2 m?

.2cm = .002m

Use a differential

$f(r) = \frac{4}{3}\pi r^3$ = volume (in m^3)

Volume of paint

$f(2+.002) - f(2) = \Delta f = \Delta y = \frac{4}{3}\pi(2.002)^3 - \frac{4}{3}\pi(2)^3$

Use a differential to approximate this

$\Delta y \approx dy = f'(2)(.002) = f'(2)dx$

$= 4\pi(2)^2(.002) \approx$

V = volume

r = radius

f(r) = V(r) above.

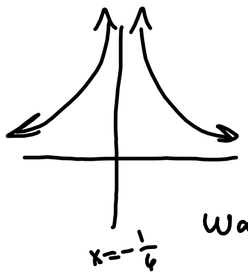
$V(2.002) - V(2) = 0.03203201$

$Vp(2) \cdot (.002) = 0.032 = V'(2)dr$

Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1. (Enter your answer using interval notation. Round your answers to three decimal places.)

$$\frac{1}{(1+6x)^4} \approx 1 - 24x$$

$x \in$ [ ]



$f(x) = (6x+1)^{-4}$

$f'(x) = -4(6x+1)^{-5}(6)$

$f'(0) = -4(1)^{-5}(6) = -24$

$L(x) = f'(0)(x-0) + f(0)$

$= -24x + 1 = L(x)$ ✓

Want $|f(x) - L(x)| < 0.1$

$\Rightarrow \left| \frac{1}{(6x+1)^4} - (-24x+1) \right| < .1$

$\Rightarrow \left| \frac{1}{(6x+1)^4} + 24x - 1 \right| < .1$

$-.1 < \frac{1}{(6x+1)^4} + 24x - 1 < .1$

$solve(g(x) - L(x) = 0.1)$

0.01846306269, $\cancel{-0.1784252247 + 0.1060473085\,I}$, -0.01511982877, -0.2673261179, $\cancel{-0.1784252247 - 0.1060473085\,I}$

$\frac{1}{(6x+1)^4} - (-24x+1)$

Blows up @ $x = -\frac{1}{6}$

$solve(g(x) - L(x) = -0.1)$

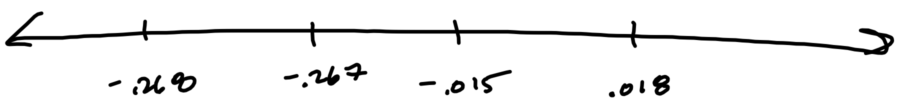
$\cancel{-0.001661616377 + 0.01654145020\,I}$, $\cancel{-0.1789442488 + 0.1067306374\,I}$, -0.2679549364, $\cancel{-0.1789442488 - 0.1067306374\,I}$, $\cancel{-0.001661616377 - 0.01654145020\,I}$

$-.26795, .018463, -.01512, -.2673$

$g(x) - L(x) < .1$

$g(x) - L(x) - .1 < 0$

$-.26795$ $-.01512$ $.018463$

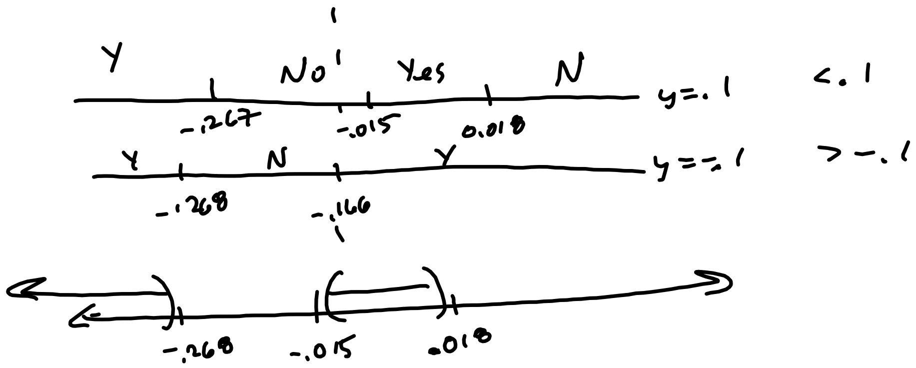


$solve(experiment(x) < .1)$

$$(-\infty, -0.2673261179), (-0.01511982877, 0.01846306269)$$

$solve(experiment(x) > -.1)$

$$(-0.2679549364, -0.1666666667), (-0.1666666667, \infty)$$



Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1. (Enter your answer using interval notation. Round your answers to three decimal places.)

$$\frac{1}{(1+6x)^4} \approx 1 - 24x$$

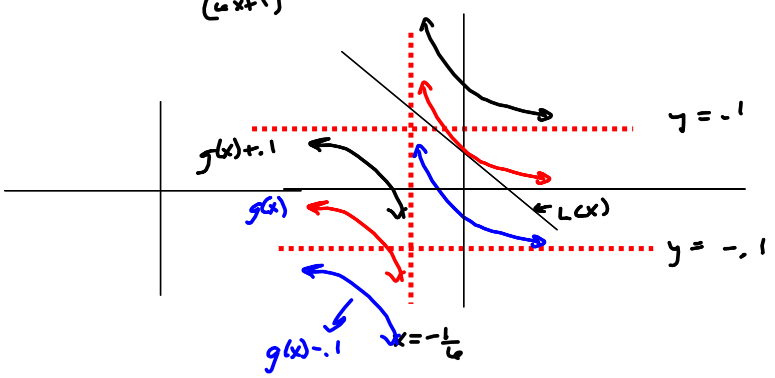
$x \in$

want $\left| \frac{1}{(6x+1)^4} - (1-24x) \right| < .1$

want $-.1 < \frac{1}{(6x+1)^4} + 24x - 1 < .1$

$\Rightarrow -.1 - \frac{1}{(6x+1)^4} < 24x - 1 < .1 - \frac{1}{(6x+1)^4}$

$\Rightarrow \frac{1}{(6x+1)^4} + 1 > -24x + 1 > \frac{1}{(6x+1)^4} - .1$



Just looking at this picture, it's evident that the tangent line never hits the g(x) + 0.1. We need the 2 intersections with g(x) - 0.1

$solve(L(x) = g(x) - .1)$

$0.01846306269, \cancel{-0.1784252247 + 0.1060473085\,I}, -0.01511982877, -0.2673261179, \cancel{-0.1784252247 - 0.1060473085\,I}$

Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1. (Enter your answer using interval notation. Round your answers to three decimal places.)

$$\frac{1}{(1+6x)^4} \approx 1 - 24x$$

$x \in \boxed{(-.015, .018)}$

$solve(L(x) = g(x) - .1)$

$0.01846306269,$ $\cancel{-0.1784252247 + 0.1060473085\,I}$, -0.01511982877, $\underline{-0.2673261179}$, $\cancel{-0.1784252247}$ $\cancel{0.1060473085\,I}$

left of asymptote