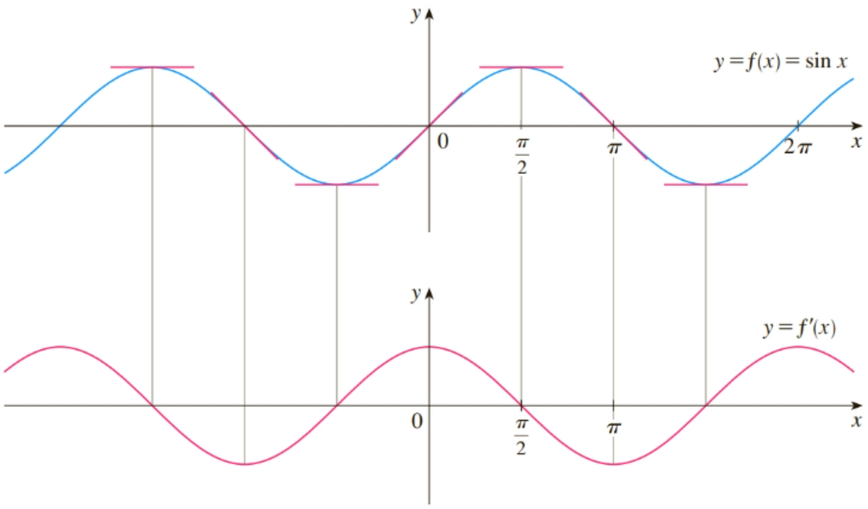
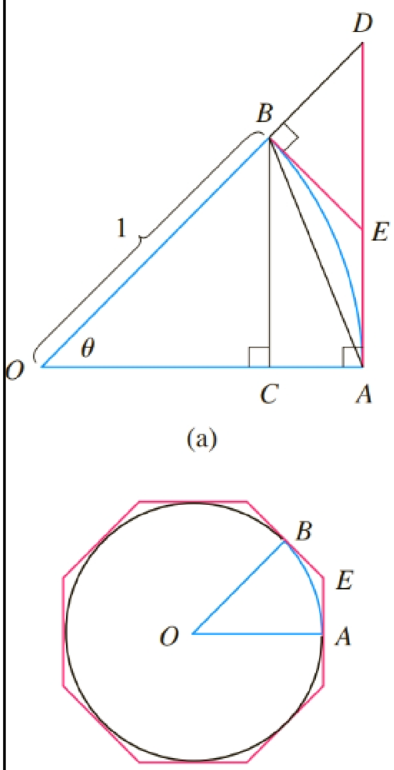
Section 2.4 Derivatives of Trig Functions

The Derivative of sine is cosine.





(a)

(b)

FIGURE 2

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

$$= \frac{\sin(x)[\cos(h)-1]+\sin(h)\cos(x)}{h}$$

$$= \sin(x)\frac{\cos(h)-1}{h} + \frac{\sin(h)}{h}\cos(x) \xrightarrow{h\to 0} \cos(x)$$

$\to 0$ as $h \to 0$

$\to 1$ as $h \to 0$

Radius is 1.

Arc length $= r\theta = \theta$

$\sin\theta = \frac{|BC|}{1} = |\overline{BC}| < |\overline{AB}| < |\text{Arc } AB| = \theta \Rightarrow$

$\frac{\sin\theta}{\theta} < 1$

Also

$\theta = |\text{Arc } AB| < |\overline{AE}| + |\overline{EB}| < |\overline{AD}| = \frac{|\overline{AD}|}{1} = \tan\theta$

i.e. $\theta < \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \cos\theta < \frac{\sin\theta}{\theta}$

This gives
$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Now take $\lim\limits_{\theta\to 0}$ (Above) :

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$\implies \quad 1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$\implies \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad \blacksquare$$

Now we need $\lim\limits_{h\to 0}\dfrac{\cos(h)-1}{h} = 0$:

$$\frac{\cos(h)-1}{h} = \frac{\cos(h)-1}{h}\cdot\frac{(\cos(h)+1)}{(\cos(h)+1)}$$

$$= \frac{\cos^2(h)-1}{(\cos(h)+1)h} = \frac{1-\sin^2(h)-1}{h(\cos(h)+1)} = \frac{-\sin^2(h)}{h(\cos(h)+1)}$$

$$= \frac{-\sin(h)}{h}\cdot\frac{\sin(h)}{\cos(h)+1} \xrightarrow{h\to 0} -1\cdot 0 = 0\ !$$

Using: $\lim(fg) = (\lim f)(\lim g)$, provided $\lim f$ & $\lim g$ $\exists$.

$$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$$

Pf
$$\frac{\cos(x+h)-\cos(x)}{h} = \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$$

$$= \frac{\cos(x)(\cos(h)-1)-\sin(x)\sin(h)}{h} = \cos(x)\left(\frac{\cos(h)-1}{h}\right)-\sin(x)\left(\frac{\sin(h)}{h}\right)$$

$$\xrightarrow{h\to 0} (\cos(x))(0)-\sin(x)(1) = -\sin(x) \blacksquare$$

$$\frac{d}{dx}\left[\tan(x)\right] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)}$$

$$\frac{f'g-fg'}{g^2}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$\boxed{\sec^2(x) = \frac{d}{dx}\left[\tan(x)\right]}$$

We now have derivatives for $\sin(x)$, $\cos(x)$, $\tan(x)$.
You can derive the derivatives of $\sec(x)$, $\csc(x)$, $\cot(x)$

$$\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$$

$$\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$$

$$\frac{d}{dx}\left[\csc(x)\right] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$

$$= \frac{d}{dx}\left[(\sin(x))^{-1}\right]$$

$$= -1(\sin(x))^{-2}(\cos(x))$$

$$= -\frac{\cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)}$$

$$= -\csc(x)\cot(x)$$

$$\frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{0\cdot\sin(x)-1(\cos(x))}{\sin^2(x)}$$

$$= -\frac{\cos(x)}{\sin^2(x)}$$

$\frac{d}{dx}[x^2 \sin(x)] = 2x\sin(x) + x^2\cos(x)$

$(fg)' = f'g + fg'$

$\frac{d}{d\theta}[\sec\theta\tan\theta] = (\sec\theta\tan\theta)\tan\theta + (\sec\theta)(\sec^2\theta)$

$f'g + fg'$

$= \boxed{\sec\theta\tan^2\theta + \sec^3\theta \quad \text{Good}}$

$f = \sec\theta \qquad g = \tan\theta$

$f' = \sec\theta\tan\theta \qquad g' = \sec^2\theta$

$= \sec\theta(\sec^2\theta - 1) + \sec^3\theta$

$= \sec^3\theta - \sec\theta + \sec^3\theta$

$= 2\sec^3\theta - \sec\theta$

$= \sec\theta(2\sec^2\theta - 1)$ etc.

Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -5$, and let $g(x) = f(x)\sin(x)$ and $h(x) = \cos(x)/f(x)$. Find the following.

(a) $g'(\pi/3)$

✖ $\frac{4-5\sqrt{3}}{2}$

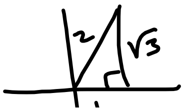
(b) $h'(\pi/3)$

✖ $\frac{5-4\sqrt{3}}{32}$

$g(x) = f(x)\sin(x)$

$\Rightarrow g'(x) = f'(x)\sin(x) + f(x)\cos(x)$

$\Rightarrow g'(\frac{\pi}{3}) = (-5)(\sin(\frac{\pi}{3})) + (4)(\cos(\frac{\pi}{3}))$



$= (-5)(\frac{\sqrt{3}}{2}) + 4(\frac{1}{2})$

$= \underline{-5\sqrt{3} + 4}$

For what values of x does the graph of f have a horizontal tangent? (Use n as your integer variable. Enter your answers as a comma-separated list.)

Find $x \ni f'(x) = 0$

$f(x) = x - 2\sin(x)$

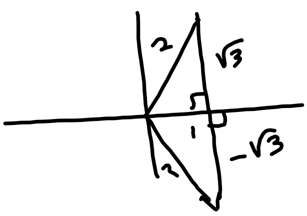
$f'(x) = 1 - 2\cos(x) \overset{SET}{=} 0 \Rightarrow$

$-2\cos(x) = -1 \Rightarrow$

$\cos(x) = \frac{-1}{-2} = \frac{1}{2}$

in $[0, 2\pi]$, we have

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ (or $60°, 300°$)



$\cos(x)$ is period 2π

$\frac{\pi}{3} + 2\pi n \quad \forall n \in \mathbb{Z}$

$= \{x \mid x \text{ is integer}\}$

Solution set:

$A = \{ \frac{\pi}{3} + 2\pi n \mid n \in \mathbb{Z} \}$

$B = \{ \frac{5\pi}{3} + 2\pi n \mid n \in \mathbb{Z} \}$

$\frac{5\pi}{3} + 2n\pi, \ n \in \mathbb{Z}$

$S' =$ Sol'n Set is $A \cup B$

$S' = \{ x + 2\pi n \mid x = \frac{\pi}{3}, \frac{5\pi}{3} \text{ and } n \in \mathbb{Z} \}$

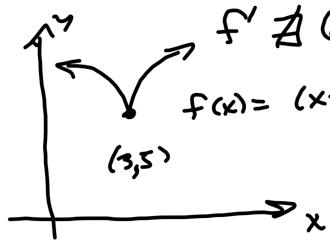
$= \{ x + 2\pi n \mid x \in \{ \frac{\pi}{3}, \frac{5\pi}{3} \} \ \ldots \ldots \}$

Horizontal and undefined tangents are targets for max/min values of f.

$f' = 0$ @ $x = 7, 11$

$f' \nexists$ @ $x = 3$

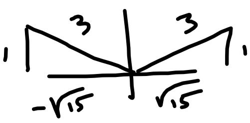
$f(x) = (x-3)^{2/3} + 5$

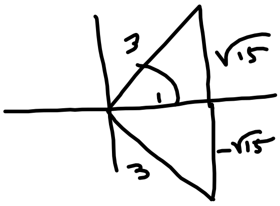


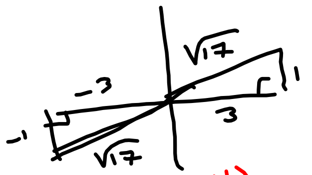
Solve on $[0, 2\pi)$

Find $x \in [0, 2\pi)$ that solve the eq'n.

$\sin(x) = \frac{1}{3}$ $\cos(x) = \frac{1}{3}$ $\tan(x) = \frac{1}{3}$







$x = \arcsin\left(\frac{1}{3}\right), \pi - \arcsin\left(\frac{1}{3}\right)$

FOR ALL REAL SOL'NS $x = \arccos\left(\frac{1}{3}\right)$, $2\pi - \arccos\left(\frac{1}{3}\right)$

$x = \arctan\left(\frac{1}{3}\right)$, $\pi + \arctan\left(\frac{1}{3}\right)$

Sol'n set

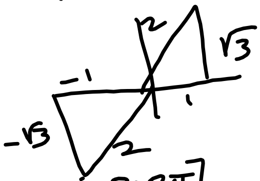
$\{x \mid \ldots + 2\pi n\}$

$\sin(x) = \frac{1}{2}$

$\{x + 2\pi n \mid x = \frac{\pi}{6}, \frac{5\pi}{6}, n \in \mathbb{Z}\}$

$\cos(x) = \frac{1}{2}$ (Above)

$\tan(x) = \sqrt{3}$



on $[0, 2\pi]$,

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

FOR ALL SOL'NS, you have a more compressed answer, b/c the solutions are π radians apart

$\left\{ \frac{\pi}{3} + \pi n \mid n \in \mathbb{Z} \right\}$

is all of 'em