Remind me to hit 'record.'

Today: Examples from 2.3 and maybe get to 2.4 Proof of "Derivative of Sine is Cosine."

17. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

$f = x^2+4x+3$

$g = x^{\frac{1}{2}}$

Want $\dfrac{f'g - fg'}{g^2}$

$\dfrac{x^2}{x^{\frac{1}{2}}} + \dfrac{4x}{x^{\frac{1}{2}}} + \dfrac{3}{x^{\frac{1}{2}}} = x^{3/2} + 4x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$\Rightarrow y' = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

$$\frac{(2x+4)(x^{\frac{1}{2}}) - (x^2+4x+3)(\frac{1}{2}x^{-\frac{1}{2}})}{(x^{\frac{1}{2}})^2}$$

$$= \frac{2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - [\frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}}]}{x}$$

$$= \frac{2x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}}{x}$$

$= \cancel{2x^{\frac{3}{2}-1} + 4x^{\frac{1}{2}-1}}$

$$= \frac{\frac{3}{2}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}}{x^1}$$

$= \frac{3}{2}x^{\frac{3}{2}-1} + 2x^{\frac{1}{2}-1} - \frac{3}{2}x^{-\frac{1}{2}-1}$

$= \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

$\Rightarrow y' = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

19. $G(q) = (1 + q^{-1})^2 = 1 + 2q^{-1} + (q^{-1})^2 = 1 + 2q^{-1} + q^{-2}$

$\Rightarrow \boxed{G'(q) = -2q^{-2} - 2q^{-3}}$

$(1+q^{-1})^2 = (1+q^{-1})(1+q^{-1}) = fg$

$\Rightarrow (fg)' = f'g + fg' = (-q^{-2})(1+q^{-1}) + (1+q^{-1})(-q^{-2})$

$= -q^{-2} - q^{-3} - q^{-2} - q^{-3} = \boxed{-2q^{-2} - 2q^{-3}}$

Cheat from 2.5

$\frac{d}{dx}[f(x)^n] = n f(x)^{n-1} \cdot f'(x)$

$G(q) = (1+q^{-1})^2 \Rightarrow$

$G'(q) = 2(1+q^{-1})(-q^{-2})$

$= 2(-q^{-2} - q^{-3})$

32. $y = \dfrac{1}{t^3 + 2t^2 - 1}$ $\Rightarrow y' = \dfrac{f'g - fg'}{g^2} = \dfrac{0(t^3+2t^2-1) - 1(3t^2+4t)}{(t^3+2t^2-1)^2}$

$f = 1$

$g = t^3+2t^2-1$

$= \dfrac{-3t^2-4t}{(t^3+2t^2-1)^2}$

Cheat

$y = (t^3+2t^2-1)^{-1}$

$\Rightarrow y' = -1(t^3+2t^2-1)^{-2}(3t^2+4t) = -\dfrac{3t^2+4t}{(t^3+2t^2-1)^2}$

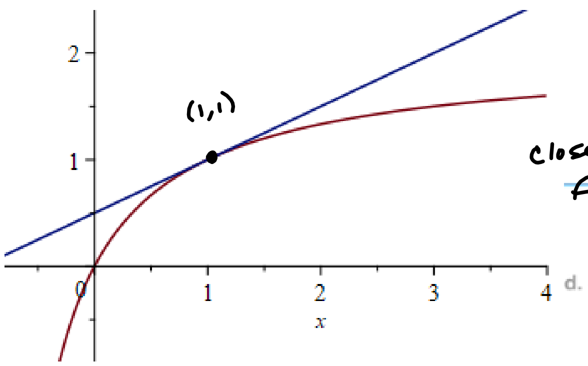
51–52 Find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{2x}{x+1}$, $(1, 1)$

$$y'\Big|_{x=1} = \frac{2(x+1) - 2x(1)}{(x+1)^2}\Bigg|_{x=1} = \frac{2(2)-2}{2^2} = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{1}{2}(x-1)+1 = \frac{1}{2}x - \frac{1}{2} + 1 = \boxed{\frac{1}{2}x + \frac{1}{2} = L(x)}$$

$L(x) =$ "Linearization of $f(x)$."



close to $x=1$,
$f(x)$ is close to tangent line!

69. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

70. Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$.
(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{f(x) + g(x)}$

69

(5,1)

(5,-3)

$(fg)'(5) = f'(5)g(5) + f(5)g'(5)$
$= (6)(-3) + (1)(2) = -18+2 = -16$

81. Find equations of both lines that are tangent to the curve $y = x^3 - 3x^2 + 3x - 3$ and are parallel to the line $3x - y = 15$.

$m = -\frac{A}{B} = \frac{-3}{-1} = 3 = m_{tan}$

$-y = -3x + 15$

$y = 3x - 15$

$y' = 3x^2 - 6x + 3 \overset{\text{SET}}{=} 3$

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x-2) = 0$

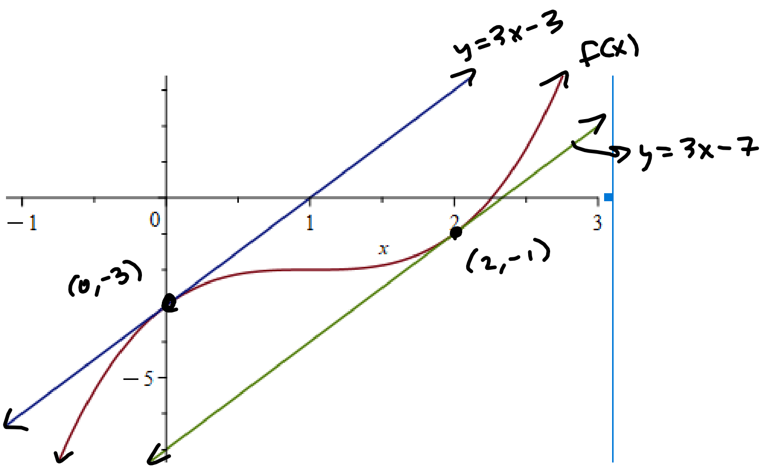
$\Rightarrow x = 0, 2$

$y(0) = -3 \rightsquigarrow (0,-3)$

$$\begin{array}{r|rrrr} 2 & 1 & -3 & +3 & -3 \\ & & 2 & -2 & 2 \\ \hline & 1 & -1 & 1 & \boxed{-1 = y(2)} \end{array} \to (2,-1)$$

$y = 3(x-0) - 3 = \boxed{3x - 3 = y}$

$y = 3(x-2) - 1 = 3x - 6 - 1 = \boxed{3x - 7 = y}$



98. At what numbers is the following function g differentiable

$$g(x) = \begin{cases} 2x & \text{if } x \leqslant 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geqslant 2 \end{cases}$$

$$g'(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2-2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$$

$\left.\begin{array}{l}\lim_{x \to 0^-} g'(x) = 2 \\ \lim_{x \to 0^+} g'(x) = 2\end{array}\right\}$ Yes! Diff'ble @ 0

$\left.\begin{array}{l}\lim_{x \to 2^-} g'(x) = -2 \\ \lim_{x \to 2^+} g'(x) = -1\end{array}\right\}$ No!

$$\boxed{\mathbb{R} \setminus \{2\}}$$

$g'(0)$, if it exists, is this

Not Perfect

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = \frac{d}{dx}[2x] = 2$$

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = \frac{d}{dx}[2x - x^2] = 2 - 2x \Big|_{x=0} = 2$$

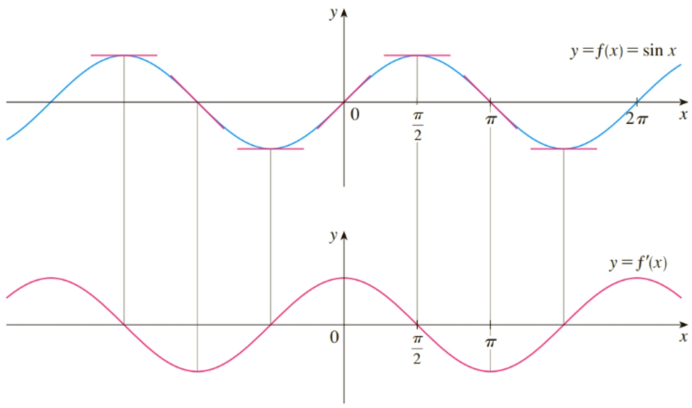
$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{2(0+h) - 0}{h} = \lim_{h \to 0^-} \frac{0+2h}{h} = \lim_{h \to 0^-} 2 = 2$$

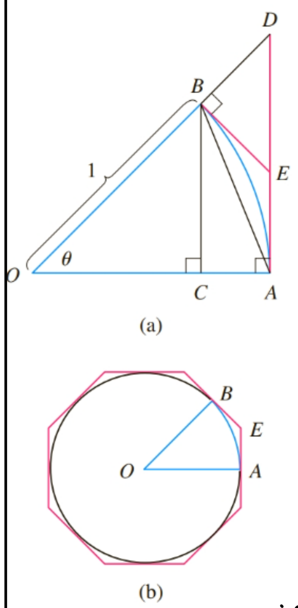
$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{2(0+h) - 2(0+h)^2 - 0}{h} = \lim_{h \to 0^+} \frac{2h - 2h^2}{h} = \lim_{h \to 0^+} \frac{h(2-2h)}{h}$$

$$= \lim_{h \to 0^+} (2 - 2h) = 2 \checkmark$$

Section 2.4 Derivatives of Trig Functions

The Derivative of sine is cosine.





(a)

(b)

FIGURE 2

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

$$= \frac{\sin(x)[\cos(h)-1]+\sin(h)\cos(x)}{h}$$

$$= \sin(x)\underbrace{\frac{\cos(h)-1}{h}}_{\to 0 \text{ as } h\to 0} + \underbrace{\frac{\sin(h)}{h}}_{\to 1 \text{ as } h\to 0}\cos(x)$$

Radius is 1.

Arc length $= r\theta = \theta$

$\sin\theta = \frac{|BC|}{1} = |\overline{BC}| < |\overline{AB}| < |\text{Arc } AB| = \theta \Rightarrow$

$\frac{\sin\theta}{\theta} < 1$

Also

$\theta = |\text{Arc } AB| < |\overline{AE}| + |\overline{EB}| < |\overline{AD}| = \frac{|\overline{AD}|}{1} = \tan\theta$

i.e. $\theta < \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \cos\theta < \frac{\sin\theta}{\theta}$

This gives

$\cos\theta < \frac{\sin\theta}{\theta} < 1$

Now take $\lim_{\theta\to 0}$ (Above):

$$\lim_{\theta\to 0}\cos\theta \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$\Rightarrow 1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$$

$$\rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \ \blacksquare$$

Now we need $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$:

$$\frac{\cos(h)-1}{h} = \frac{\cos(h)-1}{h}\cdot\frac{(\cos(h)+1)}{(\cos(h)+1)}$$

$$= \frac{\cos^2(h)-1}{(\cos(h)+1)h} = \frac{1-\sin^2(h)-1}{h(\cos(h)+1)} = -\frac{\sin^2(h)}{h(\cos(h)+1)}$$

$$= -\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{\cos(h)+1} \xrightarrow{h\to 0} -1\cdot 0 = 0\ !$$

Using: $\lim(fg) = (\lim f)(\lim g)$, provided $\lim f$ & $\lim g$ $\exists$.