1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

In many respects, Continuity's a Domain Question.

Explain why the function is discontinuous at the given number a. (Select all that apply.)

$$f(x) = \begin{cases} \dfrac{x^2 - 4x}{x^2 - 16} & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases} \qquad a = 4$$

- ☐ $f(4)$ is undefined.
- ☐ $\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4^-} f(x)$ are finite, but are not equal.
- ☐ $\lim_{x \to 4} f(x)$ does not exist.
- ☐ $f(4)$ is defined and $\lim_{x \to 4} f(x)$ is finite, but they are not equal.
- ☐ none of the above

$$\frac{x^2-4x}{x^2-16} = \frac{x(x-4)}{(x+4)(x-4)}$$

→ f(x) looks like $\frac{x}{x+4}$ everywhere but $x=4$

$$= \frac{x}{x+4}, \quad x \neq 4 \quad \xrightarrow{x \to 4} \quad \frac{4}{4+4} = \frac{4}{8} = \frac{1}{2}$$

This discontinuity is removable. Define

$$f(x) = \begin{cases} \dfrac{x^2-4x}{x^2-16} & \text{if } x \neq 4 \\ \dfrac{1}{2} & \text{if } x = 4 \end{cases}$$

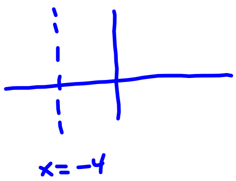
Sketch the graph of the function.

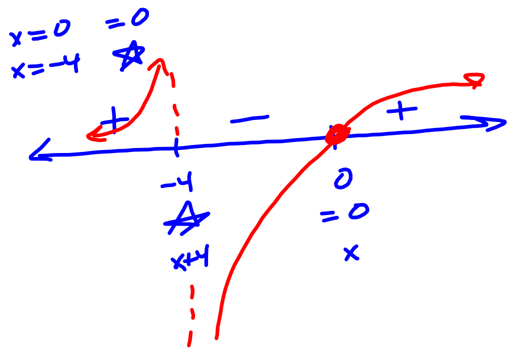
Graph $g(x) = \frac{x}{x+4}$ with a hole @ $x=4$

Then fill in $(4,1)$ spot.

$\frac{x}{x+4}$ $\quad D: \mathbb{R} \setminus \{-4\}$

V.A. $x = -4$





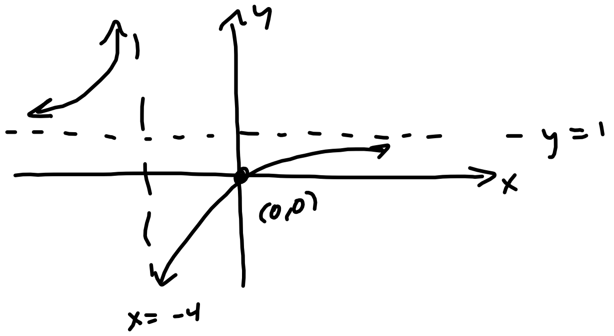
End behavior

$$\frac{x}{x+4} \quad \xrightarrow{x \to \pm\infty} \quad \frac{x}{x} = 1$$

$$\frac{x}{x(1+\frac{4}{x})} = \frac{1}{1+\frac{4}{x}} \quad \xrightarrow{x \to \pm\infty} \quad \frac{1}{1} = 1$$

$\frac{4}{x} \to 0$

$y = 1$ is Horizontal Asymptote



Claim $\lim_{x\to 2}(x^2-3x+5) = 3$

LINES

$\delta = \frac{\varepsilon}{\text{slope}}$

Scratch: want $\delta > 0 \ni$

$|x^2-3x+5-3| < \varepsilon$

$|(x-2)(x-1)| = |x-2||x-1|$

$|x^2-3x+2| = \underbrace{|x-2|}_{<\delta}|x-1|$

Need a bound on this!

Assume we're "close" to $x=2$. Then, say, within 1 unit.

That means $\delta \le 1$

$1 \le x \le 3$ (within 1 unit of 2!)

$0 \le x-1 \le 2$

i.e. $|x-1| \le 2$

So if we're less than 1 unit from $x=2$

then $|x-2||x-1| \le (|x-2|)(2) = 2|x-2| < \varepsilon \Longrightarrow$

$|x-2| < \frac{\varepsilon}{2}$

Proof Let $\varepsilon > 0$. Define $\delta = \min\{1, \frac{\varepsilon}{2}\}$. Then

$0 < |x-2| < \delta \Longrightarrow |f(x)-3| = |x^2-3x+5-3| = |x^2-3x+2|$

$= |x-1||x-2| < 2|x-2| < 2\delta \le 2\cdot\frac{\varepsilon}{2} = \varepsilon$ ∎

Claim: $\lim_{x \to 7} (3-5x) = -32$

Proof:
Let $\varepsilon > 0$ be given. Define $\delta = \frac{\varepsilon}{5}$. Then if $0 < |x-7| < \delta$, we have

$$|3-5x-(-32)| = |-5x+35| = |5x-35|$$

$$= 5|x-7| < 5\delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon \blacksquare$$

Bonus: Prove $\lim_{x \to 2} (x^3 - 5x^2 + 1) = -11$ $\to x-2$

scratch:

$|x^3 - 5x^2 + 1 - 11| = |x^3 - 5x^2 - 10|$ $\to +12!$ $\to -(-11)$ silly!

$x^3 - 5x^2 + 0x - 10$

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 0 & -10 \\ & & 2 & -6 & -12 \\ \hline & 1 & -3 & -6 & \end{array}$$

Meh

$$|x^3 - 5x^2 + 1 - (-11)| = |x^3 - 5x^2 + 12|$$

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 0 & 12 \\ & & 2 & -6 & -12 \\ \hline & 1 & -3 & -6 & 0 \\ & x^2 & x & c & r \end{array}$$

So $x^3 - 5x^2 + 12 = \underbrace{(x-2)}_{<\delta}\underline{(x^2 - 3x - 6)}$

Need a bound on this in the neighborhood of $x=2$

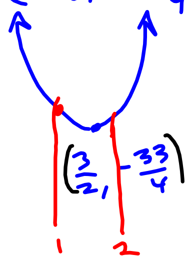
Assume $\delta \le 1$

Then $1 \le x \le 3$

Look @ x^2-3x-6 on $[1,3]$

$x^2-3x+\left(\frac{3}{2}\right)^2-\frac{9}{4}-\frac{6}{1}\cdot\frac{4}{4}$

$= \left(x-\frac{3}{2}\right)^2-\frac{33}{4}$



Plug in $x=1$

$1^2-3(1)-6=-8 \rightsquigarrow |-8|=8$

$x=3$

$3^2-3(3)-6=-6 \rightsquigarrow |6|$

$\frac{33}{4}=9+\frac{1}{4}=9.25$

$\left|-\frac{33}{4}\right|=9.25$ BIGGEST!

Write Proof

Let $\varepsilon>0$ be given. Define $\delta=\{1, \frac{\varepsilon}{9.25}\}$.

Then $0<|x-2|<\delta \Rightarrow |x^3-5x^2+1-(-11)|$

$=|x^3-5x^2+12| = |x-2|\,|x^2-3x-6| < |x-2|\,9.25$

$< 9.25\,\delta \le 9.25\cdot\frac{\varepsilon}{9.25} = \varepsilon$ ■