| | |
|---|---|
| Ch 1: Functions and Limits | 9 |
| 1.1: Four Ways to Represent a Function | 10 |
| 1.2: Mathematical Models: A Catalog of Essential Functions | 23 |
| 1.3: New Functions from Old Functions | 36 |
| **1.4: The Tangent and Velocity Problems** | **45** |
| 1.5: The Limit of a Function | 50 |
| 1.6: Calculating Limits Using the Limit Laws | 62 |
| 1.7: The Precise Definition of a Limit | 72 |
| 1.8: Continuity | 82 |

Evaluate the limit, if it exists. (If an answer does not exist, enter DNE.)

#16

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} =$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

$f(x) = x^3$.

Find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ = etc.

We're finding the derivative of the function by the limit definition.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \xrightarrow{h \to 0} 3x^2.$$

$(h \neq 0)$

1.6 #17

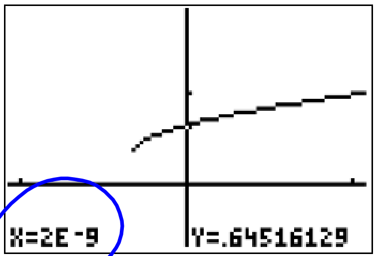
A graphing calculator is recommended.

(a) Estimate the value of $\lim_{x \to 0} \frac{x}{\sqrt{1 + 3x} - 1}$ by graphing the function $f(x) = \frac{x}{\left(\sqrt{1 + 3x} - 1\right)}$.

(b) Make a table of values of $f(x)$ for $x$ close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

We'll do it graphically, and then do it algebraically.




Almost zero

Note $f(0) = \frac{0}{\sqrt{1+0} - 1} = \frac{0}{1-1} = \frac{0}{0}$ is undefined

```
         .6666716667
Y1(-.00001
         .6666616667
Y1(-.0000001
         .6666666222
Y1(.00000001
         .6666666667
```

Numerical suggests it's approaching $y = 2/3$!

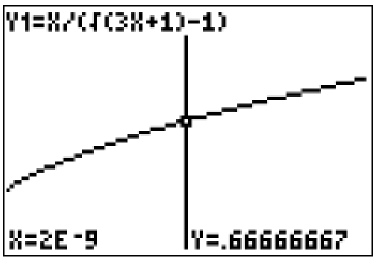
.6666666666666.....

(c) Algebraically

$$\frac{x}{\sqrt{3x+1} - 1} = \left(\frac{x}{\sqrt{3x+1} - 1}\right)\left(\frac{\sqrt{3x+1} + 1}{\sqrt{3x+1} + 1}\right)$$

$$= \frac{x\left(\sqrt{3x+1} + 1\right)}{3x+1-1} = \frac{x\left(\sqrt{3x+1} + 1\right)}{3x} = \frac{\sqrt{3x+1} + 1}{3} \quad x \neq 0$$

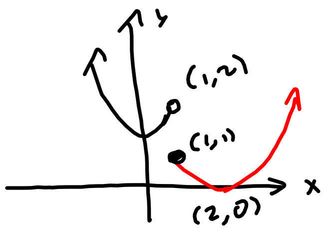
$$\xrightarrow{x \to 0} \frac{\sqrt{3(0)+1} + 1}{3} = \frac{\sqrt{1} + 1}{3} = \frac{1+1}{3} = \boxed{\frac{2}{3} = \lim_{x \to 0} f(x)}$$




After a couple of ZOOM-ins, we got darned close!

Let 1.6 #24

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}.$$




$$\lim_{x\to 1^-} f(x) = 2$$

$$\lim_{x\to 1^+} f(x) = 1$$

$$\lim_{x\to 1} f(x) \nexists$$

Find the limit, if it exists. (If an answer does not exist, enter DNE.)

#21

$$\lim_{x\to 0.2^-} \frac{5x-1}{|5x^3 - x^2|}$$

$$f(x) = \begin{cases} \frac{5x-1}{5x^3-x^2} & \text{if } 5x^3-x^2 \geq 0 \\ -\left(\frac{5x-1}{5x^3-x^2}\right) & \text{if } 5x^3-x^2 < 0 \end{cases} = \begin{cases} \frac{5x-1}{x^2(5x-1)} & \text{if } 5x^3-x^2 \geq 0 \\ -\frac{5x-1}{x^2(5x-1)} & \text{"} \quad \text{"} < 0 \end{cases}$$

$$5x^3 - x^2 = x^2(5x-1)$$

$$= \begin{cases} \frac{1}{x^2} & \text{if } 5x^3-x^2 > 0 \\ -\frac{1}{x^2} & \text{if } 5x^3-x^2 < 0 \end{cases}$$

$-$ $-$ $+$

$0$ $\frac{1}{5}$

$x^2$ $\nexists$ $5x-1$ $\nexists$

$$= \begin{cases} \frac{1}{x^2} & \text{if } x > \frac{1}{5} \\ -\frac{1}{x^2} & \text{if } x < \frac{1}{5} \end{cases}$$

FROM THE RIGHT $\frac{5x-1}{x^2(5x-1)} = \underset{x \neq \frac{1}{5}}{\frac{1}{x^2}} \xrightarrow{x \to \frac{1}{5}} \frac{1}{\left(\frac{1}{5}\right)^2} = 25$

FROM THE LEFT $-\frac{5x-1}{x^2(5x-1)} = -\underset{x \neq \frac{1}{5}}{\frac{1}{x^2}} \xrightarrow{x \to \frac{1}{5}} \frac{-1}{\left(\frac{1}{5}\right)^2} = -25$

#18

A graphing calculator is recommended.

Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \cos(10\pi x) = 0$.

Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos(10\pi x)$, and $h(x) = x^2$ on the same screen.

Let $f(x) = -x^2$, $g(x) = x^2 \cos(10\pi x)$, and $h(x) = x^2$. Then [? ] $\le \cos(10\pi x) \le$ [? ] $\Rightarrow$ [? ] $\le x^2 \cos(10\pi x) \le$ [? ]. Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) =$ [ ], by the Squeeze Theorem we have $\lim_{x \to 0} g(x) =$ [ ].

If $f(x) \le g(x) \le h(x)$

Then $\lim_{x \to c} f(x) \le \lim_{x \to c} g(x) \le \lim_{x \to c} h(x)$ & if

$L = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$

Prove $\lim_{x \to 0} x^2 \cos(10\pi x) = 0$ In a minute

Prove $\lim_{x \to 0} x^2 \cos\left(\frac{\pi}{x}\right) = 0$

$-1 \le \cos \frac{\pi}{x} \le 1$ for all $x \ne 0$

$-x^2 \le x^2 \cos\left(\frac{\pi}{x}\right) \le x^2$

$\xrightarrow{x \to 0} 0$ $\xrightarrow{x \to 0} 0$

Therefore $x^2 \cos\left(\frac{\pi}{x}\right) \xrightarrow{x \to 0} 0$ !