MAT 2410 Fall, 2023 Test 3 Take-Home: Graphing

1. (15 pts) Sketch the graph of the trigonometric polynomial $g(x) = 2\sin(x) + \cos(2x)$ on the interval $[0, 2\pi]$, showing all intercepts, extremes and inflection points. Your graphs must capture the essence of the shape, especially the concavity features. I want to see your work, with *exact* values and 4-decimal-place approximations for all x- and y-values in your legend.

$\mathcal{D} = \mathbb{R}$

$\boxed{\text{y-int: } (0,1)}$

x-int: $2\sin(x) + \cos(2x) = 0$

$\Rightarrow 2\sin(x) + 1 - 2\sin^2(x) = -2\sin^2(x) + 2\sin(x) + 1 = 0 \Rightarrow$

$2\sin^2(x) - 2\sin(x) - 1 = 0 \Rightarrow$

$2u^2 - 2u - 1 = 0 \Rightarrow$

$u^2 - u - \frac{1}{2} = 0 \Rightarrow$

$u^2 - u = \frac{1}{2} \Rightarrow$

$u^2 - u + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \left(u - \frac{1}{2}\right)^2 = \frac{3}{4} \Rightarrow$

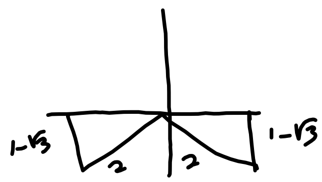
$u = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = \frac{1 \pm \sqrt{3}}{2} = \sin(x)$

$g'(x) = 2\cos(x) - 2\sin(2x)$

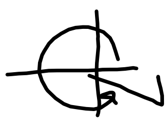
$g''(x) = -2\sin(x) - 4 + 8\sin^2(x)$

$\Rightarrow \sin(x) = \frac{1-\sqrt{3}}{2}$ OR $\sin(x) = \frac{1+\sqrt{3}}{2} \approx \frac{2.73}{2} > 1$

Nevah!

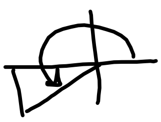


$\Rightarrow x = 2\pi + \arcsin\left(\frac{1-\sqrt{3}}{2}\right)$



≈ 5.908450875 Q $\underline{\text{IV}}$

or $\pi - \arcsin\left(\frac{1-\sqrt{3}}{2}\right)$



≈ 3.516327087 Q $\underline{\text{III}}$

Note $\frac{1-\sqrt{3}}{2} < 0 \Rightarrow$

$\arcsin\left(\frac{1-\sqrt{3}}{2}\right) < 0$, so

When we add, we're subtracting.

$\boxed{\begin{array}{l}\text{x-int:} \\ (5.9085, 0) \\ (3.5163, 0)\end{array}}$

$g(x) = 2\sin(x) + \cos(2x) \Rightarrow$

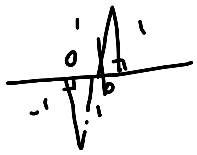
$\boxed{g'(x) = 2\cos(x) - 2\sin(2x)} = 2\cos(x) - 4\sin(x)\cos(x) \overset{\text{SET}}{=} 0 \Rightarrow$

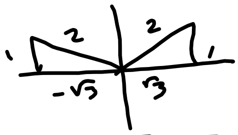
$2\cos(x)\left(1 - 2\sin(x)\right) = 0 \Rightarrow$

$\cos(x) = 0$ OR $1 - 2\sin(x) = 0$

$2\sin(x) = 1$

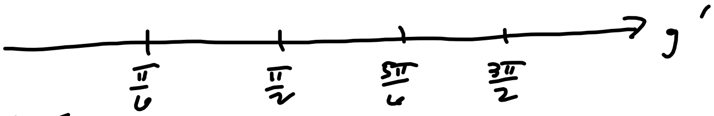
$\sin(x) = \frac{1}{2}$





$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

$g' = 0$ @ $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$



INTERVAL TEST

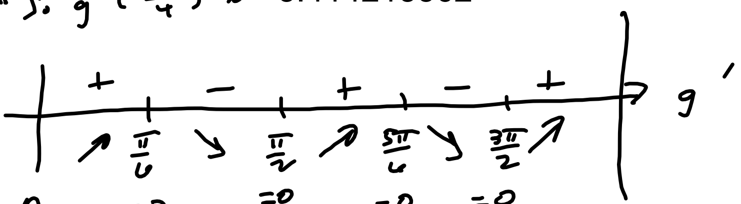
$[0, \frac{\pi}{6})$: $g'(\frac{\pi}{12}) \approx$ 0.931851653

$(\frac{\pi}{6}, \frac{\pi}{2})$: $g'(\frac{\pi}{3}) \approx$ -0.732050808

$(\frac{\pi}{2}, \frac{5\pi}{6})$: $g'(\frac{2\pi}{3}) \approx$ 0.732050808

$(\frac{5\pi}{6}, \frac{3\pi}{2})$: $g'(\pi) = -2$

$(\frac{3\pi}{2}, 2\pi]$: $g'(\frac{7\pi}{4}) \approx$ 3.414213562



=0 MAX $(\frac{\pi}{6}, \frac{3}{2})$ =0 MIN $(\frac{\pi}{2}, 1)$ =0 MAX $(\frac{5\pi}{6}, \frac{3}{2})$ =0 MIN $(\frac{3\pi}{2}, -3)$

(.5326, 1.5000) (1.5708, 1.0000) (2.6180, 1.5000) (4.7124, -3.0000)

$g''(x) = -2\sin(x) - 4\cos(2x) = -2\sin(x) - 4(1 - 2\sin^2(x))$

$g''(x) = -2\sin(x) - 4 + 8\sin^2(x) \overset{SET}{=} 0 \Rightarrow$

$\Rightarrow 8\sin^2(x) - 2\sin(x) - 4 = 0 \Rightarrow$

$4\sin^2(x) - \sin(x) - 2 = 0 \Rightarrow$

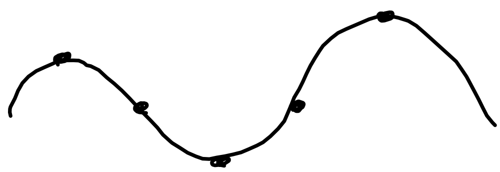
$4u^2 - u - 2 = 0 \Rightarrow$

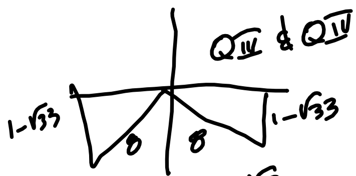
$a = 4, b = -1, c = -2 \Rightarrow$

$b^2 - 4ac = 1^2 - 4(4)(-2) = 1 + 32 = 33$

$\Rightarrow u = \frac{1 \pm \sqrt{33}}{2(4)} = \frac{1 \pm \sqrt{33}}{8} = \sin(x)$

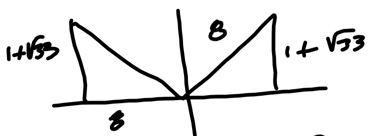
$\Rightarrow \sin(x) = \frac{1 - \sqrt{33}}{8}$ OR $\sin(x) = \frac{1 + \sqrt{33}}{8}$





QIII & QIV

$1 - \sqrt{33}$, 8



QI & QII

$1 + \sqrt{33}$, 8

$x = 2\pi + \arcsin\left(\frac{1 - \sqrt{33}}{8}\right) \approx$ 5.648318437

$\boxed{(5.6483, -.8896)\ IP}$

or $x = \pi - \arcsin\left(\frac{1 - \sqrt{33}}{8}\right) \approx$ 3.776459525

$\boxed{(3.7765, -.8896)\ IP}$

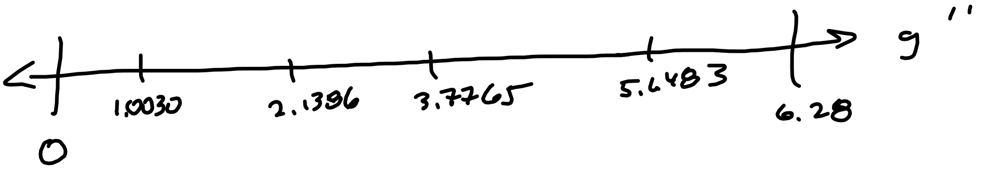
$x = \arcsin\left(\frac{1 + \sqrt{33}}{8}\right) \approx$ 1.002966954

$\boxed{(1.0030, 1.2646)\ IP}$

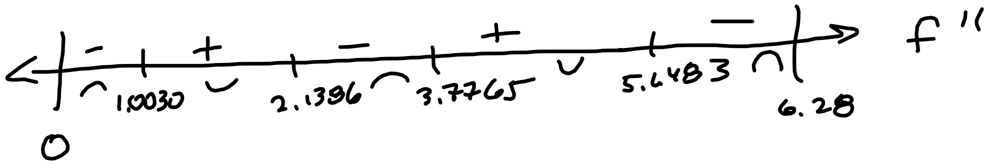
OR $x = \pi - \arcsin\left(\frac{1 + \sqrt{33}}{8}\right)$

$\approx$ 2.138625700

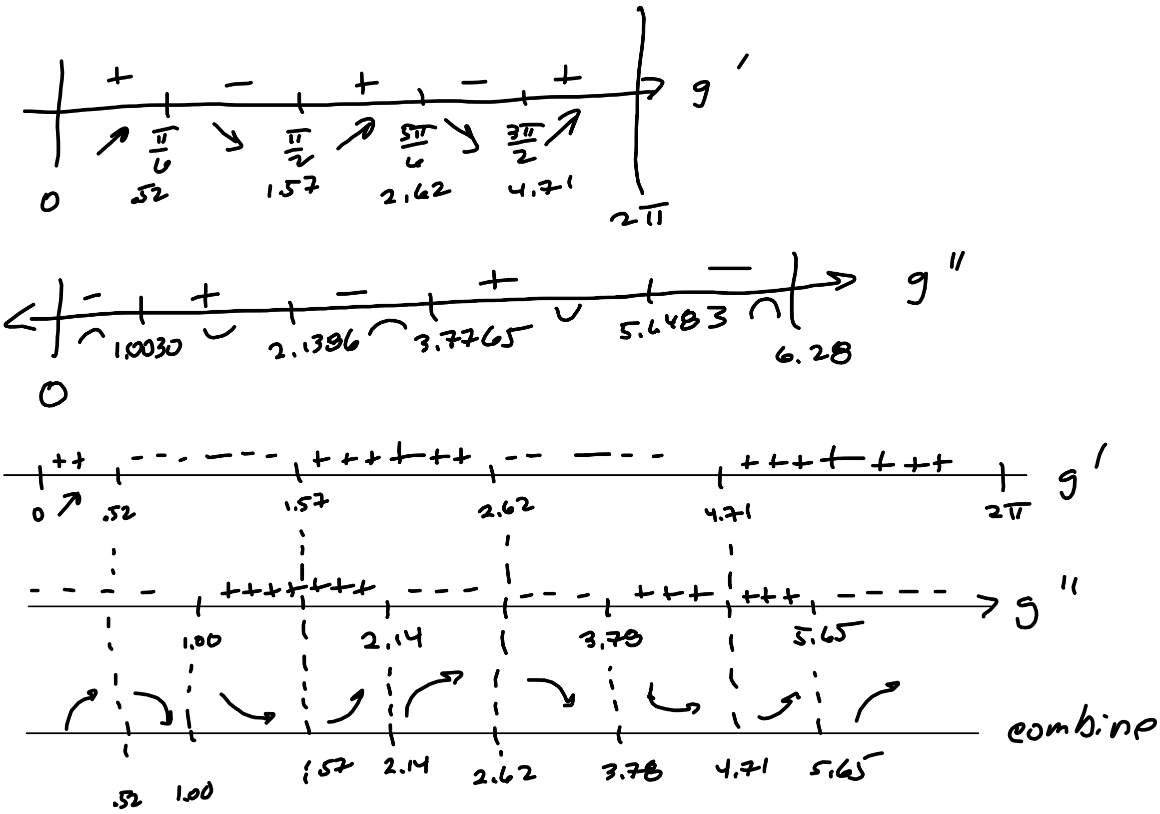
$\boxed{(2.1386, 1.2646)\ IP}$



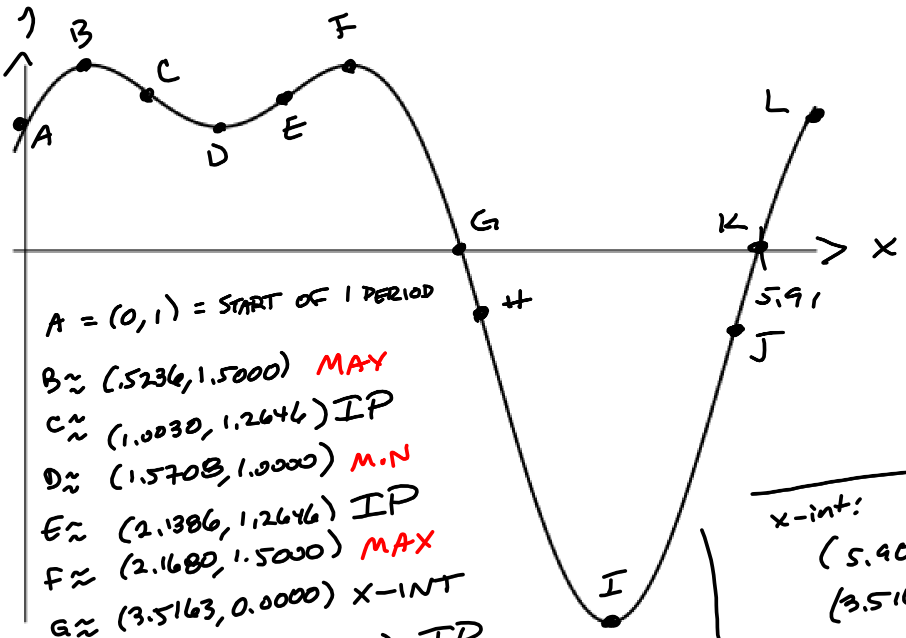
$[0, 1.0030)$	$f''(\frac{1}{2}) \approx$	-3.120060301
$(1.0030, 2.1396)$	$f''(1.5)$	1.964980013
$(2.1396, 3.7765)$	$f''(3)$	-4.122921163
$(3.7765, 5.6483)$	$f''(4)$	2.095605126
$(5.6483, 2\pi]$	$f''(6)$	-2.816584839



Now we need to place these relative to the same work done on f'.


+ − + − +
g′
π/6 π/2 5π/6 3π/2
0 .52 1.57 2.62 4.71
2π
− + − + −
g″
1.0030 2.1396 3.7765 5.6483
0
6.28
g′
0 .52 1.57 2.62 4.71 2π
g″
1.00 2.14 3.79 5.65
combine
.52 1.00 1.57 2.14 2.62 3.78 4.71 5.65

Final Answer:



$A = (0,1)$ = START OF 1 PERIOD

$B \approx (.5236, 1.5000)$ MAX

$C \approx (1.0030, 1.2646)$ IP

$D \approx (1.5708, 1.0000)$ MIN

$E \approx (2.1386, 1.2646)$ IP

$F \approx (2.1680, 1.5000)$ MAX

$G \approx (3.5163, 0.0000)$ X-INT

$H \approx (3.7765, -.8896)$ IP

$I \approx (4.7124, -3.0000)$ MIN

$J \approx (5.6483, -.8896)$ IP

$K \approx (5.9085, 0.0000)$ X-INT

$L \approx (6.2832, 1)$ = END OF 1 PERIOD

x-int:

$\boxed{(5.9085, 0)}$

$\boxed{(3.5163, 0)}$

$\boxed{(1.0030, 1.2646)}$ IP

$\boxed{(5.6483, -.8896)}$ IP

or $x = \pi - \arcsin\left(\frac{1+\sqrt{33}}{8}\right)$

$\approx$ 2.138625700

$\boxed{(2.1386, 1.2646)}$ IP

or $x = \pi - \arcsin\left(\frac{1-\sqrt{33}}{8}\right) \approx$ 3.776459525

$\boxed{(3.7765, -.8896)}$ IP

2. (15 pts) Sketch the graph of $R(x)=\dfrac{x^2-3x-28}{x-1}$. Show all intercepts, extremes, asymptotes (vertical and oblique), inflection points, and end behavior (This one has an oblique asymptote.). If you're a slave to scale, you can lose the essence of the graph's main features.

$\mathcal{D}=\mathbb{R}\setminus\{1\}$

$R(x)=0 \Rightarrow x^2-3x-28=0$

$\Rightarrow (x-7)(x+4)=0$

$\Rightarrow x=-4$ OR $x=7$

$\boxed{\text{X-INT: } (-4,0),(7,0)}$

$R(0)=\dfrac{-28}{-1}=28$

$\boxed{\text{y-INT: } (0,28)}$

$x=1$ does NOT make the numerator zero $\Rightarrow$

$\boxed{x=1 \text{ is a VERTICAL ASYMPTOTE,}}$

because $x=1$ DOES make the denominator zero

THERE IS NO HORIZONTAL ASYMPTOTE.

Degree of numerator $=2$ & Degree of denominator $=1$

$\Rightarrow$ Asymptote of degree $2-1=1$, obtained by long division:

$$
\begin{array}{r l}
 & x-2 \\
x-1 \,\big) & x^2-3x-28 \\
 & -(x^2-x) \\
\hline
 & -2x-28
\end{array}
\qquad \frac{x^2}{x}=x \qquad \frac{-2x}{x}=-2
$$

$\boxed{y=x-2 \text{ is the slant asymptote.}}$

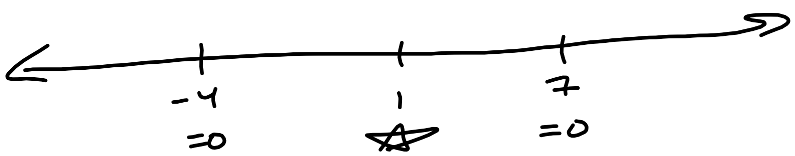
We obtain a College-Algebra graph from this info.

Cut points @ zeros of numerator and denominator.

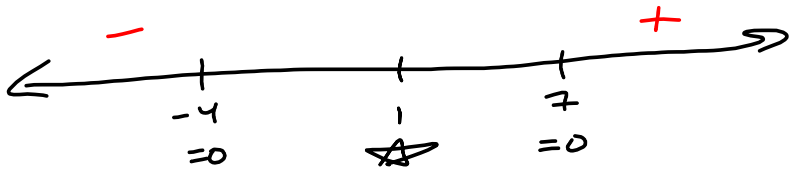
$x=1,7,-4 \;=-4,1,7$ from left to right

$x=1$ is where it blows up ☆

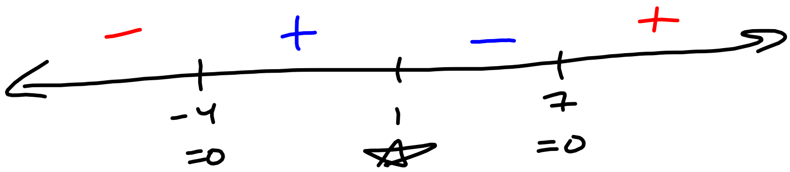
$x=-4,7$ are zeros

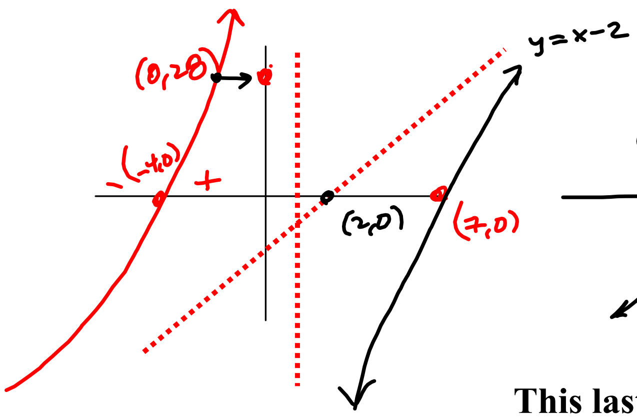


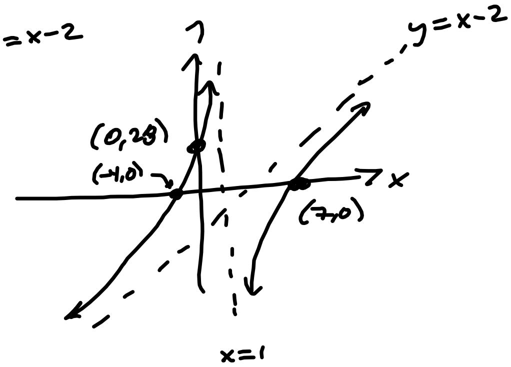
From slant asymptote $y=x-2$, we know End Behavior is ↙ ... ↗, so



$x+4$ controls @ $x=-4$.
$x+4$ changes sign.
$x-1$ controls @ $x=1$.
$x-1$ changes sign.
$x-7$ controls @ $x=7$.
$x-7$ changes sign. → This confirms our "+" from end behavior.







This last picture is about the best a college algebra student might do on this graph.

Now we do the calculus.

$$R(x) = \frac{x^2-3x-28}{x-1} = \frac{f(x)}{g(x)} \Longrightarrow$$

$$R'(x) = \frac{f'g - fg'}{g^2} = \frac{(2x-3)(x-1) - (x^2-3x-28)(1)}{(x-1)^2} = \frac{2x^2-2x-3x+3-x^2+3x}{(x-1)^2}$$

$$= \frac{x^2-2x+31}{(x-1)^2}.$$ Blows up @ $x=1$, as expected.

Now, $R'(x) \overset{SET}{=} 0 \Rightarrow x^2-2x+31=0 \Rightarrow b^2-4ac = 2^2-4(1)(31)$

$= 4-124 = -120 < 0$

No real zeros.

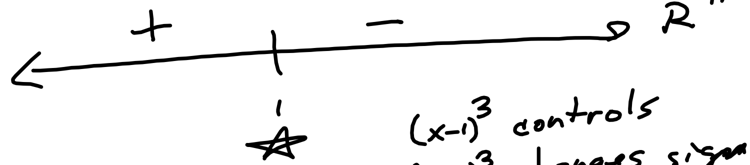
we do have a spot where it might change sign @ $x=1$, but that's controlled by the $(x-1)^2$ in the denominator, which does NOT change sign, because it's to the 2nd power, and '2' is even.

We just have to test one # & we kind of already know that $R'(x) > 0$, where it's defined. But let's check:

$$R'(x) = \frac{x^2-2x+31}{(x-1)^2} = \frac{f}{g} \Rightarrow R'' = \frac{f'g-fg'}{g^2}$$

$$= \frac{(2x-2)(x-1)^2-(x^2-2x+31)(2(x-1)')}{((x-1)^2)^2} = \frac{(x-1)\left((2x-2)(x-1)-2(x^2-2x+31)\right)}{(x-1)^4}$$

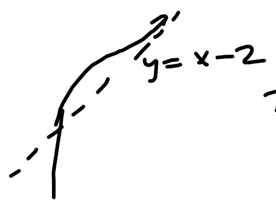
$$= \frac{2x^2-2x-2x+2-2x^2+4x-62}{(x-1)^3} = \frac{-60}{(x-1)^3}$$



$(x-1)^3$ controls
$(x-1)^3$ changes sign
Test $x=2$: $\frac{-60}{1} = -6$ "—"

It turns out that a college-algebra student, following best practices, would get exactly what a calculus student would get. But only the calculus student would KNOW that there're no little, unexpected wiggles, like you occasionally see with these graphs.

We KNOW that nothing like this happened:



To cross & then approach from above, it'd need an inflection point. We Know there's no inflection point.

I'll still be looking for your derivative and 2nd-derivative work, and look for your calculus conclusions. But getting the college-algebra graph correct is pretty much getting the whole picture.