Section 3.1 - Local and Absolute Extremes

Defs:

Absolute Max, Absolute Min.

Local Max, Local Min.

Critical number.

Closed-Interval Method for finding all local and absolute extremes on a closed interval.

Thms:

Fermat's

Extreme Value Theorem

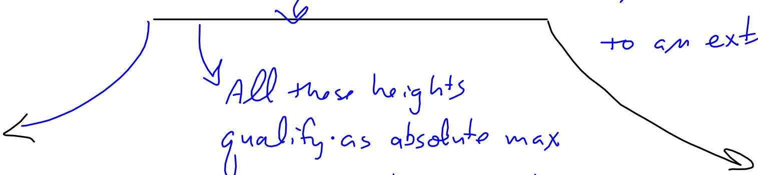
1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

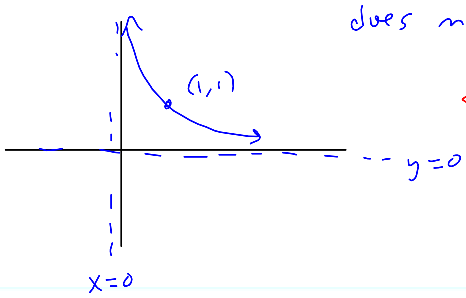
There could be more than one x-value corresponding to an extreme.



All these heights qualify as absolute max on domain shown.

3 The Extreme Value Theorem If f is $\boxed{\text{continuous}}$ on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Non example: $\frac{1}{x}$ on $(0, \infty)$ is cnt's, but unbounded, i.e., does not have an abs max / min.



unbounded

Its greatest lower bound is $y=0$, but $\frac{1}{x}=0$ has no sol'n.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

$f'(c)$ exists means $f(x)$ is continuous and differentiable at $x=c$.

Watch out. $A \Rightarrow B$ does NOT mean $B \Rightarrow A$, although we'll sorta use it that way!

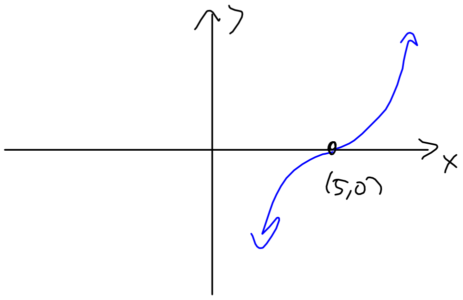
$f(c)$ is extreme & $f'(c)$ $\exists$ $\Rightarrow$ $f'(c)=0$

But $f'(c)=0$ $\not\Rightarrow$ $f(c)$ is extreme.

$f(x) = (x-5)^3 \Rightarrow$

$f'(x) = 3(x-5)^2(1) = 3(x-5)^2 \overset{SET}{=} 0 \Rightarrow$

$x=5$, i.e., $f'(5)=0$



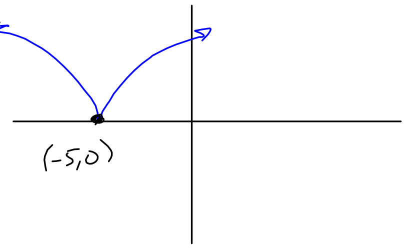
(5,0) is neither max/min point.

At the same time $f'(x) \overset{SET}{=} 0$ is a great way to find candidates

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's

$f(x) = (x+5)^{2/3}$



$f'(x) = \frac{2}{3}(x+5)^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x+5}} \neq 0$, ever!

Denom $= 0 \Rightarrow$ $3\sqrt[3]{x+5} = 0$

$\sqrt[3]{x+5} = 0$

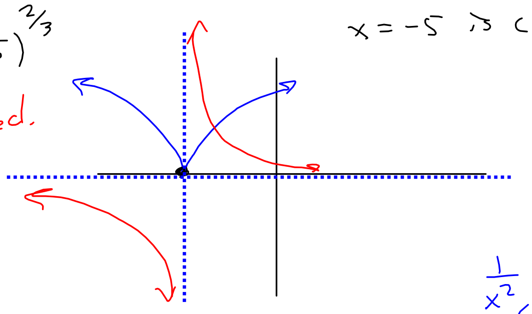
$x+5 = 0$

$x = -5$ is critical #.

$f(-5) = 0$ is absolute & local minimum.

$f(x) = (x+5)^{2/3}$

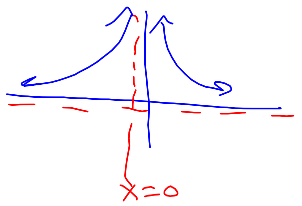
$f'(x)$ in Red.



$\frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^{1/3}}, \frac{1}{x^{1/5}} \ldots$

have same basic shape.

$\frac{1}{x^2}, \frac{1}{x^4}, \ldots, \frac{1}{x^{2/3}}, \frac{1}{x^{4/7}}, \frac{1}{x^{4/11}}$



7 If f has a local maximum or minimum at c, then c is a critical number of f.

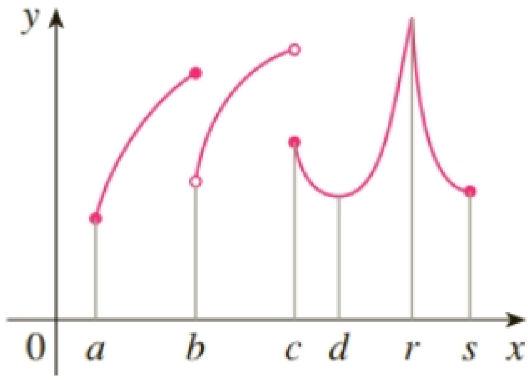
Rounds out Fermat's Thm.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

4.

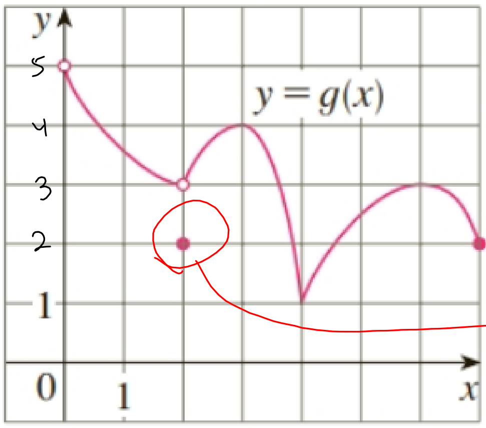


a. abs min
b. local max
c. neither
d. local & Abs.
r. Abs. Max
s. Local min.

a is an endpoint, so we can't compare it to anything to its left

5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

6.



Abs. Max: $\nexists$

Abs. Min: $y=1$

Local Max: $y = 4, 3$

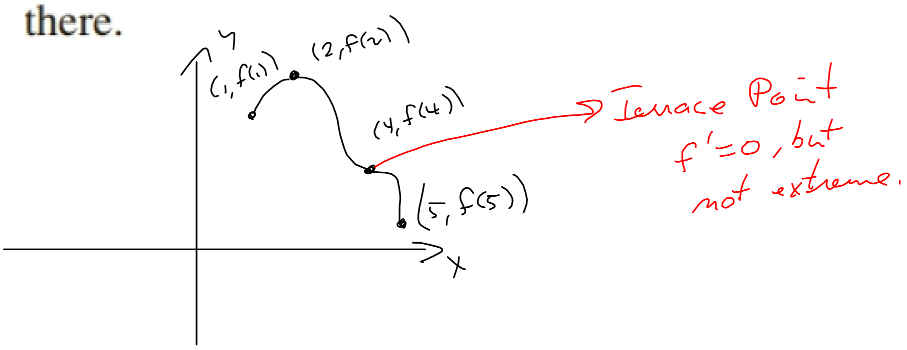
Local min: $y = 2,$

Also a local min!

7–10 Sketch the graph of a function f that is continuous on [1, 5] and has the given properties.

x= x=

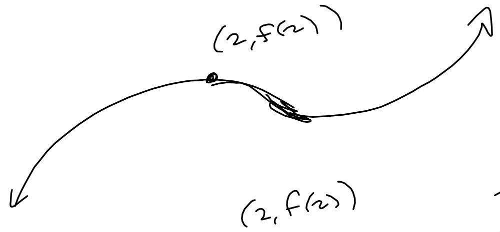
10. Absolute maximum at 2, absolute minimum at 5, 4 is a critical number but there is no local maximum or minimum there.



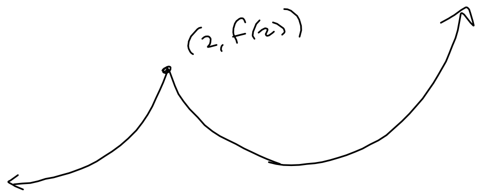
11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.

(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.

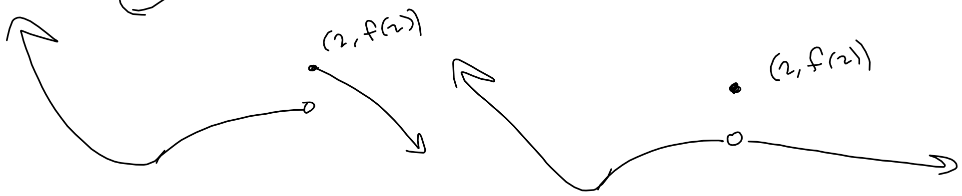
a



b



c Local Max @ x=2. Not cnt^{s} @ x=2

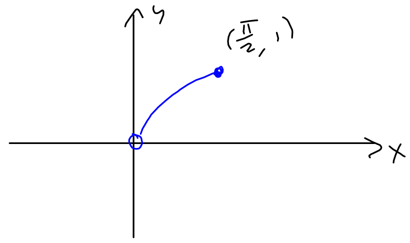


15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

Extreme

20. $f(x) = \sin x, \quad 0 < x \leqslant \pi/2$

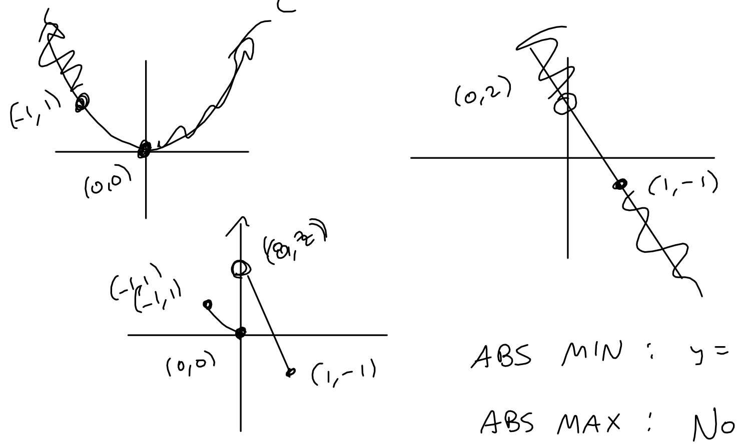
27. $f(x) = \begin{cases} x^2 & \text{if } -1 \leqslant x \leqslant 0 \\ 2 - 3x & \text{if } 0 < x \leqslant 1 \end{cases}$



No min

Abs max of 1 @ $x = \frac{\pi}{2}$

$f(x) = \begin{cases} x^2 & -1 \leq x \leq 0 \\ 2-3x & 0 < x \leq 1 \end{cases}$



ABS MIN : $y = -1$ @ $x = 1$

ABS MAX : None

LOCAL MIN : $y = 0$ @ $x = 0$.

Local MAX $y = 1$ @ $x = -1$

Find the critical #s:

All values c in the domain of f such that f ' = 0 or f ' DNE

$f(x) = x^3+6x^2-15x \Rightarrow$

$f'(x) = 3x^2+12x-15 = 3(x^2+4x-5) \overset{SET}{=} 0 \Rightarrow$

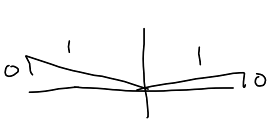
$(x+5)(x-1) = 0 \Rightarrow x = -5, 1$ are critical

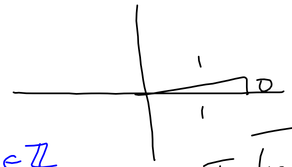
$f(\theta) = 2\cos\theta + \sin^2\theta \rightarrow$

$f'(\theta) = -2\sin\theta + 2\sin\theta\cos\theta \overset{SET}{=} 0 \Rightarrow$

$-2\sin\theta(1-\cos\theta) \Rightarrow$

$\sin\theta = 0$ or $\cos\theta = 1$

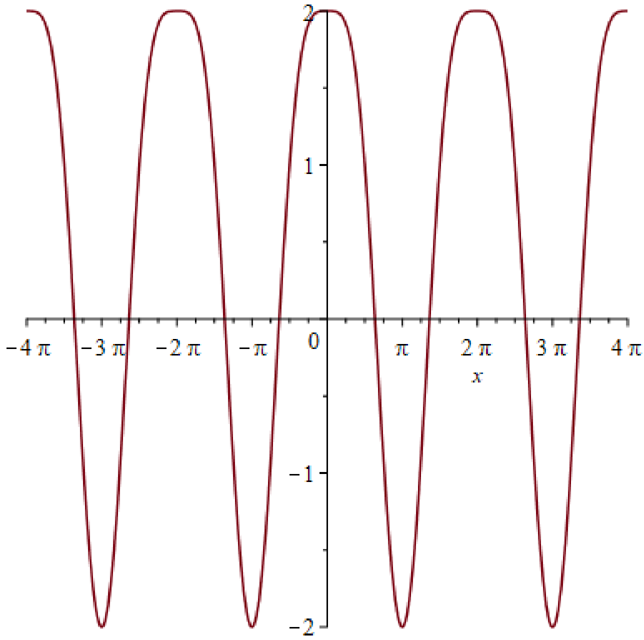




n is an integer.

$\underbrace{0+2n\pi, \ \pi+2n\pi}, \ n\in\mathbb{Z}$ $\qquad$ $2n\pi$ $\boxed{n\in\mathbb{Z}}$

WebAssign wants you to compress this down to $n\pi$



$$f(x) = x^{\frac{4}{5}}(x-4)^2 \longrightarrow$$

$$f'(x) = \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + x^{\frac{4}{5}}\left(2(x-4)'\right)$$

$$= \frac{4(x-4)^2}{5x^{1/5}} + 2x^{\frac{4}{5}}(x-4) = \frac{4(x-4)^2 + 2x^{\frac{4}{5}}(x-4)\cdot 5x^{1/5}}{5x^{\frac{1}{5}}} = 0$$

$$= \frac{2(x-4)\left[2(x-4)+5x\right]}{5x^{1/5}} = \frac{2(x-4)\left[7x-8\right]}{5x^{1/5}} \underset{=}{\text{SET}}\ 0$$

$$\boxed{\Longrightarrow x = \frac{8}{7}, 4 \qquad f' \not\exists:\ x = 0}$$

Alternate Approach:

$$\frac{x^{\frac{4}{5}}}{x^{-1/5}} = x$$

$$f'(x) = \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + x^{\frac{4}{5}}\left(2(x-4)'\right)$$

$$= 2x^{-\frac{1}{5}}(x-4)\left[\frac{2}{5}(x-4) + x\right]$$

$$= \frac{2(x-4)\left[\frac{2}{5}x - \frac{8}{5} + \frac{5}{5}x\right]}{x^{\frac{1}{5}}} = \frac{2(x-4)\left[\frac{7}{5}x - \frac{8}{5}\right]}{x^{1/5}} \underset{=}{\text{SET}}\ 0$$

$$\Longrightarrow x = \frac{8}{7}, 4$$

$$f' \not\exists \ @\ x = 0$$

The meat of it:

45–56 Find the absolute maximum and absolute minimum values of f on the given interval.

49. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$

51. $f(x) = x + \dfrac{1}{x}, \quad [0.2, 4]$

55. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

54. $f(t) = \dfrac{\sqrt{t}}{1 + t^2}, \quad [0, 2]\ 0, 3]$

51 $f(x) = x + \frac{1}{x}$ on $[0.2, 4]$:

$x + \frac{1}{x} = \frac{x^2+1}{x}$

$= x + x^{-1}$

$f'(x) = 1 + -1x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$ SET $= 0$

$\Rightarrow x = \pm 1$

$f' \nexists$ @ $x = 0 \notin$ original domain

$f(.2) = .2 + \frac{1}{.2} = .2 + \frac{1}{\frac{2}{10}} = \frac{2}{10} + 5 = \frac{26}{5} = \boxed{5.2}$

$f(4) = 4 + \frac{1}{4} = \frac{17}{4} = \boxed{4.25}$

$f(-1)$ $-1 \notin$ Domain given

$f(1) = 1 + \frac{1}{1} = 2$

min of $y = 2$ @ $x = 1$
max of $y = 5.2$ @ $x = 0.2$

64. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan\theta = \mu$.