Section 1.8

Continuity and consequences of continuity.

f is cnt^s @ c if $\lim_{x\to c} f(x) = f(c)$. NO HOLES. NO BREAKS.

(This implies that $c \in \mathcal{D}(f)$, the way I wrote it.)

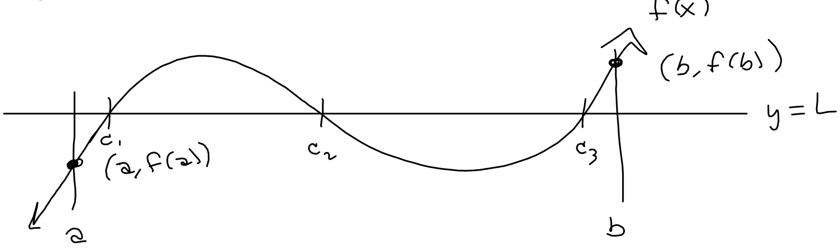
Extreme value theorem: (True, but wait 'til Chapter 3!)

f cnt^s on $[a,b] \Longrightarrow f$ achieves a max and a min somewhere in $[a,b]$.

Intermediate value Theorem:

f cnt^s on $[a,b]$ & $f(a) < L < f(b)$ (OR $f(a) > L > f(b)$)

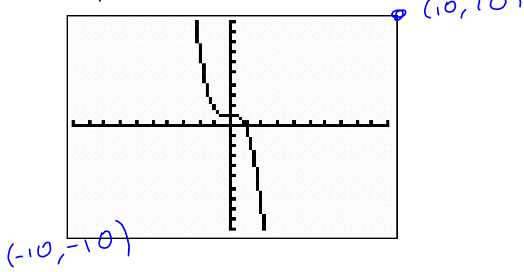
$\Longrightarrow \exists\, c \in (a,b) \ni f(c) = L$.



Prove $\cos(x) - x^3$ has at least one real zero!

" $\cos(x) = x^3$ " " " " " solution.

$\boxed{-x^3}$ End Behavior



Properties of cnt^s funcs f, g.

$f \pm g$, fg, $\frac{f}{g}$ (where $g(x) \neq 0$) are cnt^s

$f \circ g$ is cnt^s @ c, if g is cnt^s @ c and f is cnt^s at $g(c)$.

12. 0/1 points

For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 3 \\ x^3 - cx & \text{if } x \geq 3 \end{cases}$$

$c =$ ☐ ✗ 7/4

Need $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

Need $c(3)^2 + 2(3) = 3^3 - 3c$

$9c + 6 = 27 - 3c$

$12c = 21$

$c = \frac{21}{12} = \frac{7}{4} = c$

$$f(x) = \begin{cases} \frac{7}{4}x^2 + 2x & x < 3 \\ x^3 - \frac{7}{4}x & x \geq 3 \end{cases}$$

$x\left(\frac{7}{4}x + 2\right)$ $x < 3$

$\frac{7}{4}x + 2 = 0$

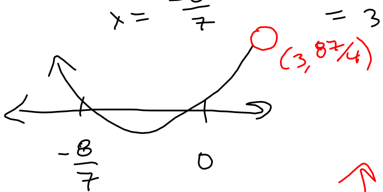
$\frac{7}{4}x = -2$

$x = \frac{-8}{7}$

$3\left(\frac{7}{4}(3) + 2\right)$

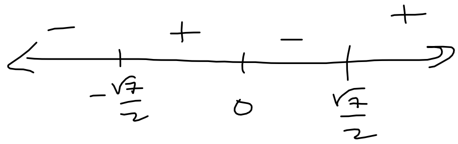
$= 3\left(\frac{21}{4} + \frac{8}{4}\right)$

$= 3\left(\frac{29}{4}\right) = \frac{87}{4}$



$$\begin{array}{r} 58 \\ 29 \\ \hline 87 \end{array}$$

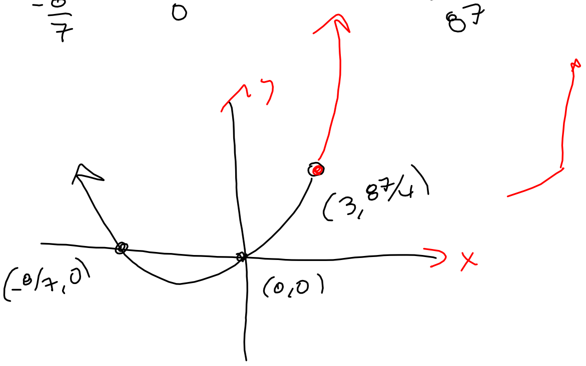
$x \geq 3$ $x\left(x^2 - \frac{7}{4}\right)$

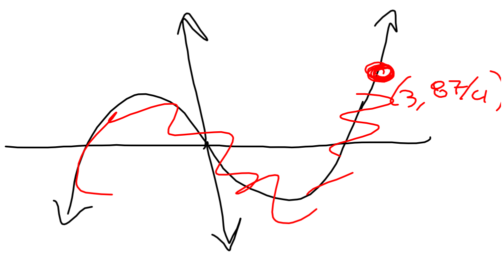


$3\left(3^2 - \frac{7}{4}\right)$

$= 3\left(9 - \frac{7}{4}\right)$

$= 3\left(\frac{36-7}{4}\right) = \left(\frac{29}{4}\right)(3) = \frac{87}{4}$





#10 Why's $Q(x)$ cnt$^{\underline{s}}$ on its domain?

$$Q(x) = \frac{\sqrt[3]{x-4}}{x^3-4}$$

Roots, quotients, sums & differences and compositions are cnt$^{\underline{s}}$ wherever they exist.

$\mathcal{D}(Q) = \mathbb{R} \setminus \{\sqrt[3]{4}\}$ Mills Answer.

$= (-\infty, \sqrt[3]{4}) \cup (\sqrt[3]{4}, \infty)$ WebAssign Answer

13. 0/1 points

Suppose f and g are continuous functions such that $g(6) = 5$ and $\lim_{x \to 6} [3f(x) + f(x)g(x)] = 48$. Find $f(6)$.

6

$g(6) = 5 = \lim_{x \to 6} g(x)$ &

$\lim_{x \to 6} f(x) = f(6)$

By continuity

$3f(x) + f(x)g(x) \xrightarrow{x \to 6} 48$

$3f(6) + f(6)(5) = 48$

$8f(6) = 48$

$f(6) = \frac{48}{8}$ $\boxed{= 6 = f(6)}$

15. 0/6 points

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$f(x) = x^4 + x - 5 = 0, \quad (1, 2)$

$f(1) = 1^4 + 1 - 5 = -3 < 0$ $\Rightarrow$ it crosses $y = 0$ line, i.e.,

implies.

$f(2) = 2^4 + 2 - 5 = 13 > 0$

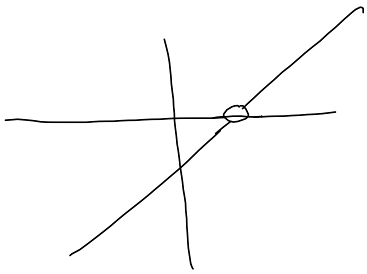
$\exists c \in (1,2) \ni f(c) = 0$

there is, in, such that

$\frac{(x-5)^2}{x-5}$ has no root in $(4,6)$

$= x-5 \quad (x \neq 5)$ Not cont's @ $x=5$ $\Rightarrow$

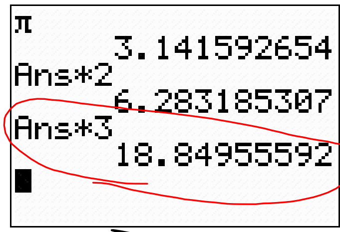
No zero @ $x=5$



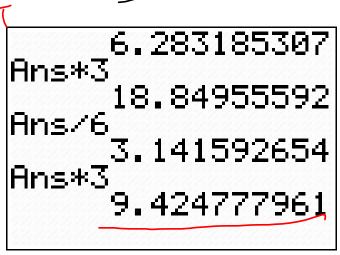
Prove $f(x) = \sin(x^3)$ has at least 2 x-intercepts in $(1,2)$

$x \in (1,2) \Rightarrow x^3 \in (1,8)$

$\sin(x) = 0 \quad \forall\, x = n\pi,\ n \in \mathbb{Z}.$



Idiot



2 places where $x^3 = n\pi$ for $x \in (1,2)$

$\therefore \sin(x^3) = 0$ in 2 spots:

$x = \sqrt[3]{\pi},\ \sqrt[3]{2\pi}$

Proof

$x^3 = \pi$ @ $x = \sqrt[3]{\pi}$ and $x^3 = 2\pi$ @ $x = \sqrt[3]{2\pi}$ $\Rightarrow$

$\sin(x^3) = 0$ twice in $(1,2)$ ∎

63. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a+h) = f(a)$$

A if and only if B means
A iff B means
$A \Longleftrightarrow B$ means
means A implies B and B implies A.

<u>Proof</u> Suppose $\lim_{h \to 0} f(a+h) = f(a)$

"$\Rightarrow$" Define $c = a+h$. Then $h \to 0 \Rightarrow x \to c$ Nahhh...

$\lim_{h \to 0} (a+h) = a$, by properties of limits.

So we're saying $\lim_{h \to 0} f(a+h) = f(a)$

means $\lim_{x \to a} f(x) = f(a) \Rightarrow$ cnt^s!

Now "$\Leftarrow$"

If f is cnt^s @ a, then $\lim_{h \to 0} f(a+h) = f(\lim_{h \to 0}(a+h))$

$= f(a) \Rightarrow \blacksquare$

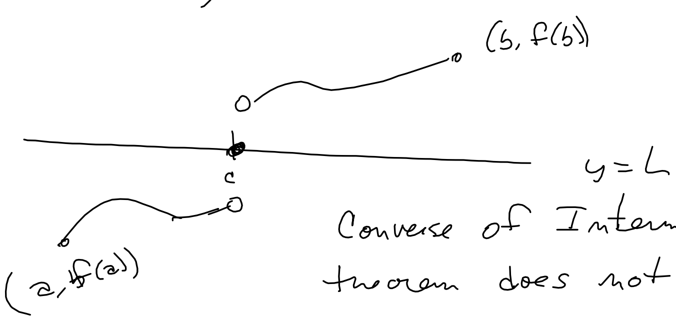
THIS IS LOGICAL EQUIVALENCE.

If it rains, I will bring an umbrella.

Bringing an umbrella does <u>NOT</u> imply it's raining.

A = I bring an umbrella.
B = It's raining.

$B \Rightarrow A$, But $A \not\Rightarrow B$.



Converse of Intermediate value theorem does not hold.

$\lim_{h \to 0} f(a+h) = f(a)$ iff f cnt^s @ $x = a$.

Prove cosine is cnt^s @ a.

Let $f(x) = \cos(x)$. Then $\cos(a+h)$

$= \cos(a)\cos(h) - \sin(a)\sin(h) \xrightarrow{h \to 0}$

$\cos(a) \cdot 1 - \sin(a)\sin(0) = \cos(a)$ ∎

(Semi-formal. The crux of it is shown.)

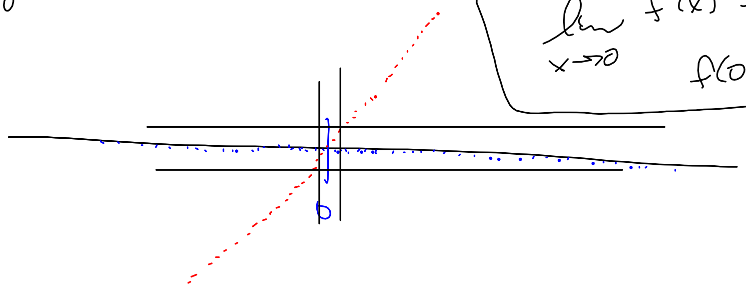
#20

Web Assign.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 5x & \text{if } x \text{ is irrational} \end{cases}$$

Mind-bending example to expose pathology &/or surprising consequences of the theory.

$$\boxed{\lim_{x \to 0} f(x) = 0 \ \& \ f(0) = 0!}$$



So f is cnt^s @ $x=0$ (and ONLY $x=0$!)

#20, §1.6 Squeeze Thm.

If $f(x) < g(x)$ on (a,b)

then $\lim_{x\to b^-} f(x) \le \lim_{x\to b^-} g(x)$

$$\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right) = 0$$

$$-1 \le \cos\left(\frac{\pi}{x}\right) \le 1$$

$$-x^2 \le x^2\cos\left(\frac{\pi}{x}\right) \le x^2$$

$$\underset{x\to 0}{\downarrow} \qquad \underset{x\to 0}{\downarrow} \qquad \underset{x\to 0}{\downarrow}$$

$$0 \qquad 0 \qquad 0$$

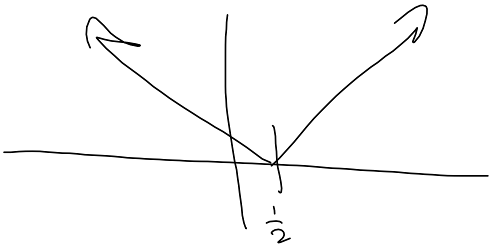
See Maple PDF from 1/25 :

https://harryzaims.com/201/201-spring-21/notes/210125-maple.pdf

$$\frac{2x-1}{|2x^3-x^2|} = \frac{2x-1}{x^2|2x-1|} = -\frac{1}{x^2} \text{ for } x<\frac{1}{2} \xrightarrow{x\to\frac{1}{2}^-} -\frac{1}{(\frac{1}{2})^2} = -4$$

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases}$$

$$|2x-1| = 2|x-\tfrac{1}{2}|$$



$$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -(2x-1) & \text{if } x < \frac{1}{2} \end{cases}$$