II 6 Properties of Exponents & Logs

b^x variable in exponent

$\log_b(b^x) = x$

"Show me your power!"

§6.1 Inverses

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{why?}$$

So you don't have to KNOW $f^{-1}(x)$ in order to do derivatives.

Like 6.1 # 18 $f(x) = x^5 + x^3 + x$ $f^{-1}(x)$ is HARD!

We know it exists, because $f'(x) = 5x^4 + 3x^2 + 1 > 0 \Rightarrow f^{-1}$ exists!

But you CAN find $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$

$f^{-1}(3)$: Solve $x^5 + x^3 + x = 3 \Rightarrow x = 1$ is a solution

the

$f^{-1}(3) = 1$

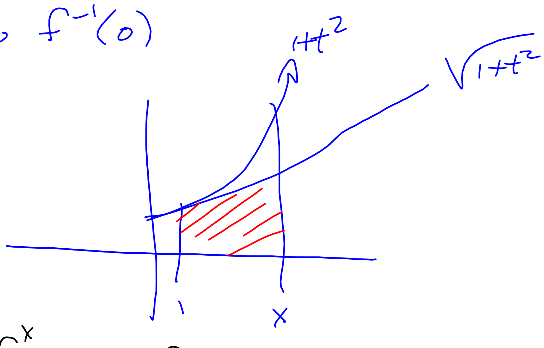
$f'(f^{-1}(3)) = f'(1) = 9$ (See f' above!)

$\therefore \boxed{(f^{-1})'(3) = \frac{1}{9}}$

6.1 # 45

$f(x) = \int_1^x \sqrt{1+t^2}\,dt$ want $(f^{-1})'(0)$

so we want to know $f^{-1}(0)$



FACTS: $\sqrt{1+t^2} \geq 0$

$f(x) = 0 \Rightarrow$

$x = 1$

$\int_1^1 \sqrt{1+t^2}\,dt = 0$

If $x > 1$, then $\int_1^x \sim > 0$.

$\therefore f^{-1}(0) = 1$!

FTC I

$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\int_1^x \sqrt{1+t^2}\,dt\right] = \sqrt{1+x^2} = f'(x) \Rightarrow f'(1) = \sqrt{2}$

$\Rightarrow (f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \boxed{\frac{1}{\sqrt{2}} = (f^{-1})'(0)}$

§6.2 Exponential Functions

If $f(x) = b^x$ for $b > 0$.

Then $f'(x) = f'(0)b^x$. Then we messed around, trying to estimate $f'(0)$ for different b's.

we want $b \ni f'(0) = 1$

$2 < b < 3 \ldots$

$b = e$ = natural base $\approx 2.718281828459045235 36$

up-shot: $\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$ Sweet!

$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$ Chain Rule

$\frac{d}{dx}\left[e^{\cos(x)}\right] = -\sin(x)e^{\cos(x)}$

MOAR 6.2

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \frac{\cancel{e^{3x}}(1-e^{-6x})}{\cancel{e^{3x}}(1+e^{-6x})} = \frac{1-0}{1+0} = \frac{1}{1} = 1$$

§6.3 Logarithms

Book $y = f(x)$ means $x = f^{-1}(y)$

$\log_b(x)$ is the inverse of b^x, so

$y = b^x$ means $x = \log_b(y)$

$y = \log_b(x)$ means $x = b^y$

$y = b^x$ Take $\log_b$ of both sides

$$\log_b(y) = \log_b(b^x) = x$$

$y = \log_b(x)$ Raise both sides to the power of b.

$$b^y = b^{\log_b(x)} = x$$

$\ln(x) = 5$

$e^{\ln(x)} = \boxed{x = e^5}$

$3^x = 7$

$\log_3(3^x) = \log_3(7)$

$x = \log_3(7) = \frac{\ln(7)}{\ln(3)}$ by change of base.
(for calculator)

Expand $\ln\left(\sqrt{\frac{x-1}{x+2}}\right) = \ln\left(\left(\frac{x-1}{x+2}\right)^{\frac{1}{2}}\right) = \frac{1}{2}\ln\left(\frac{x-1}{x+2}\right)$

$= \frac{1}{2}\left[\ln(x-1) - \ln(x+2)\right]$

$a^x a^y = a^{x+y}$, $(ab)^x = a^x b^x$, $\frac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$

$\ln(xy) = \ln(x) + \ln(y)$, $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$, $\ln(a^x) = x\ln(a)$

OLD EXAMPLE

$(16)(32) = (2^4)(2^5) = 2^9$

Multiplication by adding exponents (computing)

#18 from Book

$\ln\left(\frac{x+1}{x}\right) = 2$ $\qquad \ln(x) = \log_e(x)$

$e^{\ln(\sim)} = e^2.$

$\frac{x+1}{x} = e^2$

$x+1 = e^2 x$

$x - e^2 x = -1$

$x(1-e^2) = 1$

$\boxed{x = \frac{1}{1-e^2}}$

One of the §6.3 toughies!

#64 $f(x) = \dfrac{1-e^{-x}}{1+e^{-x}} = y$ Find $f^{-1}(x)$.

solve for y:

$$\frac{1-e^{-y}}{1+e^{-y}} = x$$

$$1-e^{-y} = x(1+e^{-y}) = x + xe^{-y}$$

$$-1 - xe^{-y} = \qquad - xe^{-y} \quad -1$$

$$-e^{-y} - xe^{-y} = x-1$$

$$-e^{-y}(1+x) = x-1$$

$$e^{-y} = \frac{x-1}{-(1+x)}$$

$$\rightarrow e^{y} = \frac{x+1}{1-x}$$

$$\ln(e^{y}) = y = \boxed{\ln\left(\frac{x+1}{1-x}\right) = f^{-1}(x)}$$

$$-y = \ln\left(\frac{1-x}{x+1}\right)$$

$$y = -\ln\left(\frac{1-x}{x+1}\right)$$

$$= \ln\left(\left(\frac{1-x}{x+1}\right)^{-1}\right)$$

$$\boxed{y = \ln\left(\frac{x+1}{1-x}\right) = f^{-1}(x)!}$$

§6.4 derivatives of logs

We made a (hurried) case for $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

Furthermore, $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$

This extends the domain for antiderivatives.

$\Longrightarrow \int \frac{dx}{x} = \ln|x| + C$ ∀ $x<0$ is OK.

Chain Rule Extensions:

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

$$\int \frac{f'(x)\,dx}{f(x)} = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$$

That's why $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{-\sin(x)}{\cos(x)}\,dx$ $\quad u = \cos(x)$, $du = -\sin(x)dx$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$

$$= \ln|(\cos(x)^{-1})| + C \mp \boxed{\ln|\sec(x)| + C = \int \tan(x)\,dx}$$

$\int e^x dx = e^x + C$, $\quad \int b^x dx = \frac{1}{\ln(b)} b^x + C$ $\Big\}$ I always got $\ln(b)$ TIMES & $\frac{1}{\ln(b)}$ confused

$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[b^x] = \ln(b)\, b^x$

So what I did was remember the idea in the derivation.

$e^{\ln(b)} = b$, so $b^x = (e^{\ln(b)})^x = e^{(\ln(b))x} \Longrightarrow$

$$\frac{d}{dx}[b^x] = (\ln(b))e^{(\ln(b))x} = (\ln(b))(e^{\ln(b)})^x = (\ln(b))\, b^x$$

So, it's "TIMES" for derivative.

THE FINAL BIT FOR MAT 201 is

Logarithmic Differentiation.

We have no way of differentiating

$f(x) = g(x)^{h(x)}$ VARIABLE BASE AND variable power!

$y = (\sin(x))^{x^2}$ The technique:

$\ln(y) = \ln\left((\sin(x))^{x^2}\right) = x^2 \ln(\sin(x))$

Now $\frac{d}{dx}$ [BOTH SIDES] $(fg)' = f'g + fg'$

$\frac{y'}{y} = 2x \ln(\sin(x)) + x^2\left(\frac{\cos(x)}{\sin(x)}\right)$

$y' = [2x \ln(\sin(x)) + x^2 \cot(x)]\, y$

$$\boxed{y' = [2x \ln(\sin(x)) + x^2 \cot(x)](\sin(x))^{x^2}}$$

$y = x^x$

$\ln y = \ln(x^x) = x\ln(x)$

$\frac{y'}{y} = 1\ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1 \Rightarrow$

$\boxed{y' = (\ln(x) + 1)\, x^x}$