Today: Questions & 6.2-6.3

Properties of Exponents & bridging to exponential functions

Let $p \in \mathbb{N} = \{1,2,3,\dots\}$.

Let $b>0$, $b \in \mathbb{R}$.

Then $b^p = \underbrace{b \cdot b \cdot b \cdots b}_{p \text{ of them}}$

$2^3 = 2\cdot 2\cdot 2$

$b^{-p} = \frac{1}{b^p}$, $b^{p+q} = b^p b^q$, $b^{p-q} = b^p b^{-q} = \frac{b^p}{b^q}$ $p, q \in \mathbb{N}$

$b^{\frac{p}{q}} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p$ when the exponent is rational.

Roots!

$4^{\frac{5}{2}} = \sqrt{4^5} = (\sqrt{4})^5 = 2^5 = 32$

You have to be a little careful with domains.

$x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2} = (\sqrt[3]{x})^2$

$x^{\frac{3}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3} = (\sqrt{x})^3$

That's not what I'm after w/ domain remark.

Most of the 055/Intermediate Algebra instruction keeps everything positive.

You need to be careful, sometimes.

$(\sqrt{x})^2 = (-5)^2$ Assumption is that $x \geq 0$, here.

$x = 25$

Be careful about domains!

Looking for the right example!

We're AIMING at b^x ($2^x, e^x, 3^x, (\frac{1}{2})^x$) defined for ALL $x \in \mathbb{R}$. Our theory really only reaches to integer & rational exponents. We don't have a satisfactory Proof for the behavior of things like

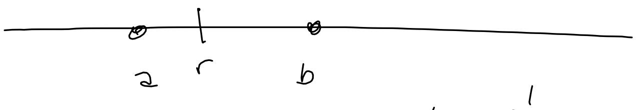
$2^{\sqrt{3}}$ OR 3^{π} OR 3^{e} ← will define "e" today.

$\sqrt{3} \approx 1.732$, so we "agree" that

$2^{1.73} < 2^{\sqrt{3}} < 2^{1.74}$, somewhere.

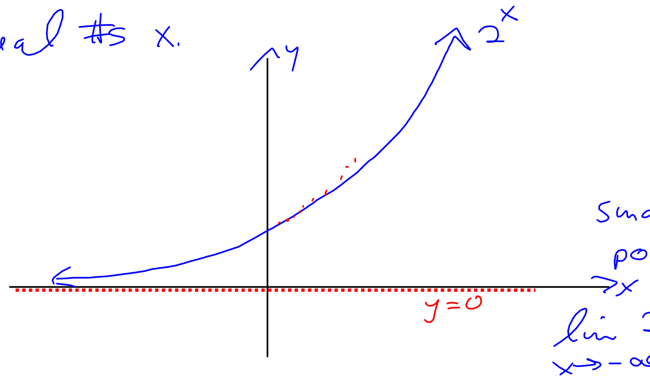
And we can go on forever, getting closer to $\sqrt{3}$ with decimals (rational)

FACT: You can always build a sequence of rational #s that converges to any irrational #. Between any 2 irrationals, there's always a rational.



Basically you can make $\frac{1}{n}$ as close to zero as you want, by taking n big enough.

b^x is a continuous function defined for all real #s x.



$2^x > 0$

$2^{-500} = \frac{1}{2^{500}}$ is small but still positive.

$\lim_{x \to -\infty} 2^x = 0$

y=0 Horizontal Asymptote.

$\lim_{x \to \infty} 2^x = \infty$

We want to be able to differentiate these functions:

$$\frac{d}{dx}[b^x] = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h-1)}{h}$$

$= \lim_{h\to 0} \left(\frac{b^h-1}{h}\right) b^x$. Trouble is, $\lim_{h\to 0} \frac{b^h-1}{h}$, b/c the b^x factors out of the limit: $\boxed{\left(\lim_{h\to 0} \frac{b^h-1}{h}\right)} b^x$

Recall, $b^0 = 1$,

$$\lim_{h\to 0} \frac{b^h - b^0}{h} = \lim_{h\to 0} \frac{b^{0+h}-b^0}{h} =$$

Derivative of b^x @ $x=0$!

$\frac{d}{dx}[b^x]$ = slope of b^x @ $x=0$ times b^x.

By numerical investigation, we find $\frac{d}{dx}[2^x]\Big|_{x=0} \approx 0.69$, so

$\frac{d}{dx}[2^x] \approx (.69)2^x$

HUGE POINT to be make here!

$\frac{d}{dx}[2^x]$ is proportional to 2^x!

Slope is proportional to height!

$y' = Ky$ for some $K \in \mathbb{R}$

Now, $\lim_{h\to 0} \frac{3^h-1}{h} \approx 1.10$

7 Definition of the Number *e*

e is the number such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

Note $\frac{2^h-1}{h} \xrightarrow{h\to 0} .69 < 1 < \lim_{h\to 0} \frac{3^h-1}{h} \approx 1.10$

we reason that $2 < e < 3$!

"e" is the natural base. "e" for "Euler"

Differential Equations

$y' - y = 0$

e^x is the solution!

$$\frac{d}{dx}[e^x] = \underbrace{\left(\lim_{h\to 0} \frac{e^h-1}{h}\right)}_{\text{Proportionality constant is } K=1!} e^x = e^x$$

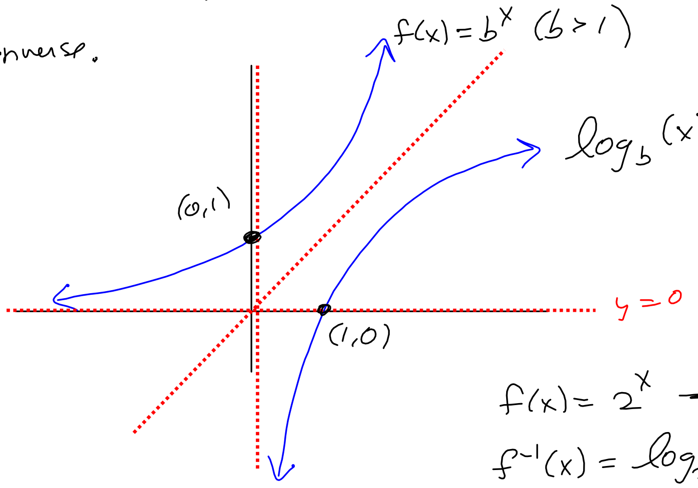
$\boxed{e \approx 2.718281828}$

2 Theorem If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $b^x > 0$ for all x. If $0 < b < 1$, $f(x) = b^x$ is a decreasing function; if $b > 1$, f is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

Now, Recall, 6.1: $y = f^{-1}(x) \longleftrightarrow x = f(y)$

Since b^x, $b > 1$ is smooth & increasing, so is its inverse.



$f(x) = 2^x \longrightarrow$

$f^{-1}(x) = \log_2(x)$

$2^{\log_2(x)} = x$

$\log_2(2^x) = x$

They're inverses, by DEFINITION of the log!

I wanna argue this:

$$\left(e^{\log_e(*)}\right) = * = e^{\ln(*)} = *$$

$$\log_e(x) = \ln(x) = \text{"Natural Logarithm"}$$

$$\frac{d}{dx}\left[e^x\right] = e^x$$

Chain Rule for it: $\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)} = e^{f(x)} \cdot f'(x)$

$$\frac{d}{dx}\left[\sin(x^2)\right] = 2x\cos(x^2) = (\cos(x^2)) \cdot 2x$$

$$\frac{d}{dx}\left[e^{x^2}\right] = 2x\,e^{x^2} = e^{x^2} \cdot 2x$$

$$\left(e^{\log_e(*)}\right) = * = e^{\ln(*)} = *$$

Bootstrap: $\frac{d}{dx}\left[2^x\right] = \frac{d}{dx}\left[\left(e^{\ln(2)}\right)^x\right] = \frac{d}{dx}\left[e^{(\ln(2))x}\right]$

$$(a^b)^c = a^{bc}$$

$$= \left(e^{(\ln(2))x}\right) \cdot \frac{d}{dx}\left[\ln(2)\,x\right]$$

$$= e^{(\ln(2))x} \cdot \ln(2) = \left(e^{\ln(2)}\right)^x \cdot \ln(2) = 2^x \cdot \ln(2)$$

slope of 2^x @ $x=0$.

The derivative of b^x is proportional to b^x & proportionality constant K is $K = \ln(b)$!

$$\frac{d}{dx}\left[b^x\right] = K b^x \quad \& \quad K = \ln(b)!$$

$$\boxed{\frac{d}{dx}\left[b^x\right] = \ln(b) \cdot b^x}$$

Change of Base : $\log_b(x) = \frac{\log_a(x)}{\log_a(b)} = \frac{\ln(x)}{\ln(b)}$

For calculators. There's no $\log_3(x)$ key.

So $\log_3(7) = \frac{\ln(7)}{\ln(3)}$

$$y = \log_a(x) \iff x = a^y$$

$$\ln(x) = \ln(a^y) = y\ln(a) \implies$$

$$\boxed{\frac{\ln(x)}{\ln(a)} = y}$$

$$\ln(x^3) = \ln(x \cdot x \cdot x) = \ln(x) + \ln(x) + \ln(x) = 3\ln(x)$$

$$a^b a^c a^d = a^{b+c+d}$$

$$\frac{d}{dx}\left[2^{x^2\sin(x)}\right] = \ln(2)\,2^{x^2\sin(x)} \cdot (2x\sin(x) + x^2\cos(x))$$

Combines a lot of it.

What about $\frac{d}{dx}[\ln(x)]$

$$y = \ln(x) \iff x = e^y$$

(I always do this; b^x & $\log_b(x)$ are inverses.

$y = \ln(x)$ Raise "e" to the power of both sides.

$e^y = e^{\ln(x)} = x$)

Now, $x = e^y$

$$\frac{d}{dx}[x] = \frac{d}{dx}[e^y] =$$

$$1 = e^y \cdot \frac{dy}{dx} \implies$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$ Sweet!

Chain Rule Version?

$$\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

$$(\ln(u))' = \frac{u'}{u}$$

Application:

$$\int \cot(x)\,dx = ?$$

$$\int \tan(x)\,dx = ?$$

$$\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{-\sin(x)}{\cos(x)}\,dx = -\int \frac{du}{u} = -\int \frac{u'(x)\,dx}{u(x)}$$

$$= -\ln(u(x)) + C!$$

$$= -\ln(\cos(x)) + C = \ln((\cos(x))^{-1}) + C$$

$$= \ln(\sec(x)) + C$$

(Actually $\ln|\sec(x)| + C$

$$\boxed{\int \sec(x)\,dx} = \int \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)}\,dx$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx = \int \frac{u'(x)\,dx}{u(x)} = \int \frac{du}{u} = \ln|u| + C$$

$$\boxed{= \ln|\sec(x) + \tan(x)| + C}$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\boxed{\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C}$$