§6.1 Inverse Functions

$f(x)$ is 1-to-1 if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

" " " " $f(x_1) = f(x_2)$ then $x_1 = x_2$ Contrapositive

$A \Rightarrow B$ is logically equivalent to Not $B \Rightarrow$ Not A

$f(x) = \frac{x-1}{x+2}$ is 1-to-1 :

\$ $f(x_1) = f(x_2)$

Then $\frac{x_1-1}{x_1+2} = \frac{x_2-1}{x_2+2}$

$\Rightarrow (x_2+2)(x_1-1) = (x_1+2)(x_2-1)$

$\Rightarrow x_1x_2 - x_2 + 2x_1 - 2 = x_1x_2 - x_1 + 2x_2 - 2$

$\Rightarrow 2x_1 - x_2 = 2x_2 - x_1$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2 \Rightarrow f$ is 1-to-1.

This property ensures that the inverse relation for f is a function.

Inverse Relation

$f = \{(1,2), (3,4), (5,6)\}$ & $f^{-1} = \{(2,1), (4,3), (6,5)\}$

$\mathcal{D} = \{1, 3, 5\}$ f is 1-to-1 $\longrightarrow$ No reps in y-coords

$\mathcal{R} = \{2, 4, 6\}$ f^{-1} is a function No reps in x-coords.

$f = \{(1,2), (2,2), (3,4)\}$ is func. but not 1-to-1.

$f^{-1} = \{(2,1), (2,2), (4,3)\}$ is RELATION, but not a function.

$f^{-1}(2)$ is not well-defined. Is it 1 or 2?

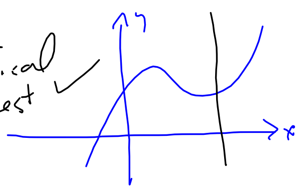
function: f is a function if each x in the domain is assigned exactly one y in the range.

f : one x means one y

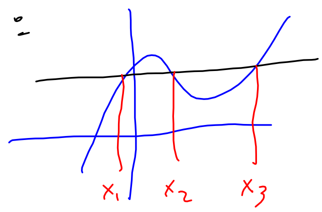
1-to-1 : one x means one y AND one y means one x.

This gives rise to the HORIZONTAL LINE TEST for 1-to-1.

func. vertical line test ✓



NOT 1-to-1 :



Fails horizontal line test.

$f(x_1) = f(x_2) = f(x_3)$!

Finding $f^{-1}(x)$:

$f(x) = x^2$ is classic:

$y = x^2$

$x = y^2$ Solve for y

$y^2 = x$

$\sqrt{y^2} = \sqrt{x}$

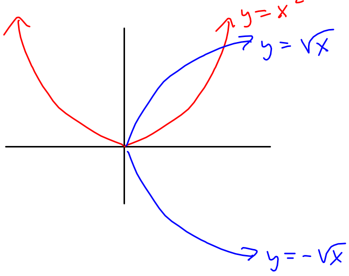
$|y| = \sqrt{x}$

$y = \pm\sqrt{x}$ NOT A Function!

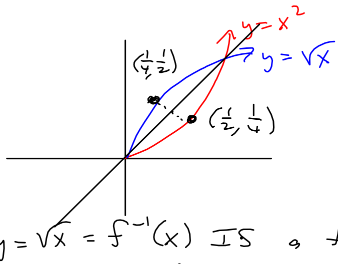
each x ($\neq 0$) gives 2 y's!

This is what we do:

Restrict Domain of $y = x^2$ to a domain on which $y = f(x)$ IS 1-to-1



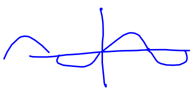
Restrict $y = x^2$ to $x \geq 0$

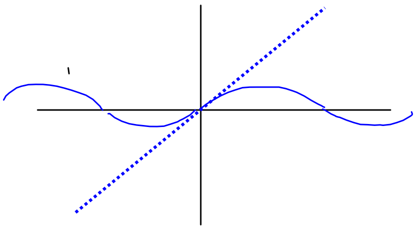


$y = \sqrt{x} = f^{-1}(x)$ IS a function!

The graph of $f(x) = x^2$ ($x \geq 0$) & $f^{-1}(x) = \sqrt{x}$ are inverse functions.

you've already done this in trig.

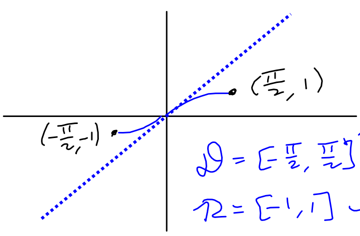




Not 1-to-1, but we want some kind of inverse!

$\sin^{-1}(x) = \arcsin(x)$

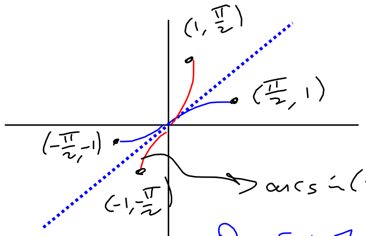
Restrict sine to where it's 1-to-1



$\mathcal{D} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\mathcal{R} = [-1, 1]$

Restricted Sine.



$\arcsin(x) = \sin^{-1}(x)$

$\mathcal{D} = [-1, 1]$

$\mathcal{R} = [-\frac{\pi}{2}, \frac{\pi}{2}]$ Arcsine

f^{-1} swaps x's & y's! So $\mathcal{D}(f) = \mathcal{R}(f^{-1})$

$\mathcal{R}(f) = \mathcal{D}(f^{-1})$

Let $f(x) = \frac{x-1}{x+2}$. Find $f^{-1}(x)$.

$x = \frac{y-1}{y+2}$ Solve for y:

$x(y+2) = xy+2y = y-1$

$\Rightarrow xy+2y-y = -1$

$xy+y = -1$

$y(x+1) = -1$

$y = \frac{-1}{x+1}$

Check:

$f \circ f^{-1} = x$ i.e. $f \circ f^{-1} =$ identity.

$f \circ f^{-1} = f(f^{-1}(x))$

$= f\left(\frac{-1}{x+1}\right) = \frac{\frac{-1}{x+1} - 1}{\frac{-1}{x+1} + 2}$

$= \frac{\frac{-1}{x+1} - \frac{1}{1}\left(\frac{x+1}{x+1}\right)}{\frac{-1}{x+1} + \frac{2}{1}\left(\frac{x+1}{x+1}\right)} = \frac{\frac{-1-(x+1)}{x+1}}{\frac{-1+2(x+1)}{x+1}}$

$= \frac{-1-x-1}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{-1+2x+2} = \frac{-2-x}{2x+1} \neq x$!

$f(f^{-1}) = \frac{f^{-1}-1}{f^{-1}+2}$

$= \frac{\left(\frac{-1}{x+1} - 1\right)(x+1)}{\left(\frac{-1}{x+1} + 2\right)(x+1)}$

$= \frac{-1-1(x+1)}{-1+2(x+1)} = \frac{-1-x-1}{-1+2x+2} = \frac{-x-2}{2x+1}$ Nope. Must've messed-up

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Try again: $x = \frac{y-1}{y+2} = x$

$y-1 = xy+2x$

$y-xy = 2x+1$

$y(1-x) = 2x+1$

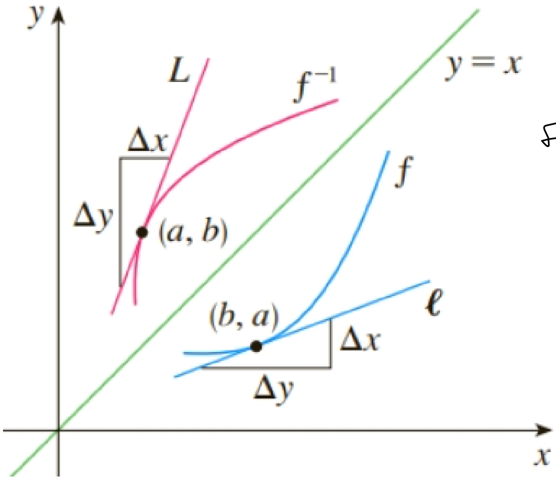
$y = \boxed{\frac{2x+1}{1-x} = f^{-1}}$?

$f \circ f^{-1} = x$?

$f(f^{-1}(x)) = \frac{f^{-1}-1}{f^{-1}+1} = \frac{\left(\frac{2x+1}{1-x} - 1\right)(1-x)}{\left(\frac{2x+1}{1-x} + 2\right)(1-x)} = \frac{2x+1-(1-x)}{2x+1+2(1-x)}$

$= \frac{2x+1-1+x}{2x+1+2-2x} = \frac{3x}{3} = x$ ✓ Finally!

Section 6.1: Visual justification for the derivative of the inverse.



Claim $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

From Picture:

$(f^{-1})'(a) \approx \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta x}{\Delta y}$ in graph of

$f(x) = \dfrac{1}{\frac{\Delta y}{\Delta x}} \approx \dfrac{1}{f'(b)} = \dfrac{1}{f'(f^{-1}(a))}$

Why do this to people?

Because f^{-1} can be DIFFICULT & TIME-CONSUMING.

$\boxed{E7}$ $f(x) = 2x + \cos(x)$. Find $(f^{-1})'(1)$

$= \dfrac{1}{f'(f^{-1}(1))}$. Need $f^{-1}(1)$ & f'.

$2x + \cos(x) = 1$

NOTE/OBSERVE $2(0) + \cos(0) = 1$ ✓

$\boxed{f^{-1}(1) = 0}$

$f'(x) = 2 - \sin(x)$

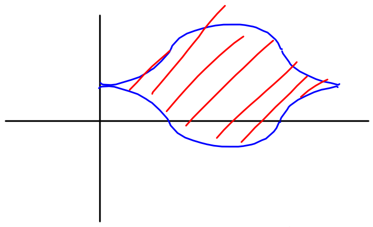
$f'(0) = f'(f^{-1}(1)) = 2 - \sin(0) = 2$

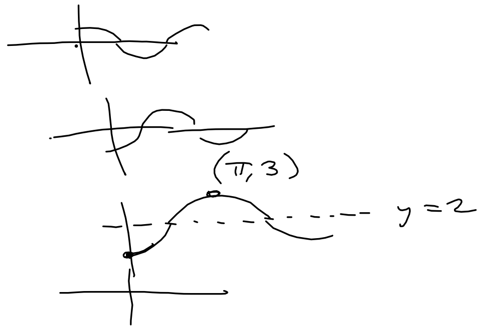
so $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \boxed{\dfrac{1}{2}}$

Tougher 5.1 question:

16 from 8th Edition. $0 \le x \le 2\pi$

$y = \cos(x)$, $y = 2 - \cos(x)$





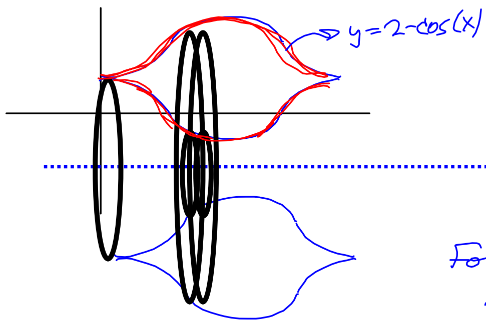
Setup Integral. $2-\cos(x)$ is always on top.

$$\int_0^{2\pi} (2 - \cos(x) - \cos(x))\,dx = \int_0^{2\pi} (2 - 2\cos(x))\,dx$$

$$= 2\int_0^{2\pi} (1 - \cos(x))\,dx = 2\Big[x - \sin(x)\Big]_0^{2\pi} = 2\big[(2\pi - 0) - (0 - 0)\big]$$

$$\boxed{= 4\pi}$$

Now, find volume of the region obtained by revolving the previous region about the Line $y=-2$



Volume by disks (actually washers)
Washer = Outer Disk - inner Disk!

$y=-2$

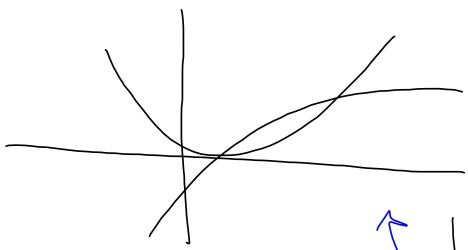
For $2-\cos(x)$, the radius is
$2-\cos(x)-(-2) = 4-\cos(x) = r_{outer}$

For $y=\cos(x)$, $r_{inner} = \cos(x)-(-2) = 2+\cos(x)$

Outer volume − inner volume = washer volume.

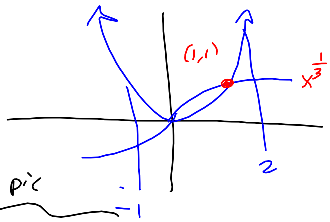
$$\int_0^{2\pi} \pi\,(4-\cos(x))^2\,dx - \int_0^{2\pi} \pi\,(2+\cos(x))^2\,dx$$

$$= \pi \int_0^{2\pi} \left((4-\cos(x))^2-(2+\cos(x))^2\right)dx = \pi \int (outer^2 - inner^2)\,dx$$



Volume of solid obtained by revolving the region bounded by

$x = -1$, $x = 2$, $y = x^2$, $y = \sqrt[3]{x}$, about $y = 5$



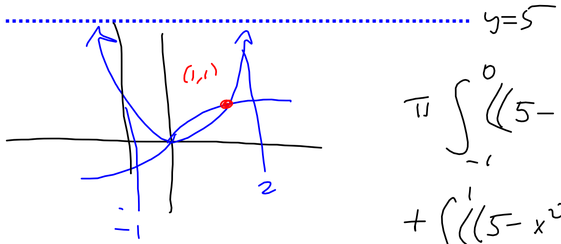
Top $\frac{1}{2}$ of pic

$$\text{Area} = \int_{-1}^{2} |x^2 - x^{\frac{1}{3}}|\,dx$$

$$= \int_{-1}^{0} (x^2 - x^{\frac{1}{3}})\,dx + \int_{0}^{1} (x^{\frac{1}{3}} - x^2)\,dx$$

$$+ \int_{1}^{2} (x^2 - x^{\frac{1}{3}})\,dx$$

Volume Question



$$\pi \int_{-1}^{0} \left((5 - x^{\frac{1}{3}})^2 - (5 - x^2)^2\right) dx$$

$$+ \int_{0}^{1} \left((5 - x^2)^2 - (5 - x^{\frac{1}{3}})^2\right) dx$$

$$+ \int_{1}^{2} \left((5 - x^{\frac{1}{3}})^2 - (5 - x^2)^2\right) dx$$ is the setup.

Poor Picture