Section 4.5 u-Substitution

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

We will show the basic idea for indefinite integrals in The Substitution Rule, above, and then give 2 strategies for definite integrals:

1. Replace the original limits with u-limits
2. Find the anti-derivative. UN--substitute. Then use the original limits.

$$\int 3\sin(3x)\,dx = \int \sin(3x)\cdot 3dx = \int \sin(u)\,du = -\cos(u)+C$$

$u = 3x$ $\quad f(g(x)) = \sin(3x)$ $\quad = -\cos(3x)+C$

$du = 3du$ $\quad g(x) = 3x$

§3.9 Differentials!

$\Rightarrow g'(x)\,dx = 3dx$

what if we didn't have that nice "3" for the $3dx = du$?

$$\int * = \frac{1}{3}\int 3*$$

$$\int \sin(3x)\,dx$$

$u =$ messy thing inside $= 3x$

then $\cancel{du =}\ 3dx = du$

$dx = \frac{du}{3}$

$$= \int \sin(u)\,\frac{du}{3} = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}\left[-\cos(u)\right] + C$$

$$= \boxed{-\frac{1}{3}\cos(3x) + C}$$

The idea is to "see" the integrand as the result of a derivative involving the chain rule.

$$\frac{d}{dx}\left[\sqrt{x^2+5}\right] = \frac{d\left[\sqrt{u}\right]}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left[u^{\frac{1}{2}}\right]\cdot\frac{du}{dx}$$

$u = x^2+5$

$\frac{du}{dx} = 2x$

$$= \left(\frac{1}{2}u^{\frac{1}{2}-1}\right)\cdot(2x)$$

$$= u^{-\frac{1}{2}}\cdot x = (x^2+5)^{-\frac{1}{2}}x = \frac{x}{\sqrt{x^2+5}}$$

$$\frac{d}{dx}\left[\sqrt{x^2+5}\right] = \frac{d}{dx}\left[(x^2+5)^{\frac{1}{2}}\right] = \frac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+5}}$$

Now, consider

$$\int \frac{x}{\sqrt{x^2+5}}\,dx = \int$$

$u =$ mess inside $= x^2+5$

$\Rightarrow du = 2x \cdot dx \quad = du$

$dx = \frac{du}{2x}$ } Student way

$$= \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= (x^2+5)^{\frac{1}{2}} + C = \sqrt{x^2+5} + C$$

Old Hand:

$$\int \frac{x}{\sqrt{x^2+5}}\,dx =$$

$u = x^2+5$ I've got the x. I lack the factor of "2"

$du = 2x\,dx$

$$\text{so } = \frac{1}{2}\int (x^2+5)^{-\frac{1}{2}}(2x\,dx) = \frac{1}{2}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{x^2+5} + C$$

The NEXT level

$$\int x^3\sqrt{x^2+5}\,dx = \int x^3\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int x^2\sqrt{u}\,du$$

$u = x^2+5$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

Remember the "$\frac{1}{2}$"!

still have an "x" in there?!

Now $u = x^2+5 \Rightarrow x^2 = u-5$, so ...

$$= \frac{1}{2}\int (u-5)\sqrt{u}\,du = \frac{1}{2}\int (u-5)u^{\frac{1}{2}}\,du = \frac{1}{2}\int\left(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 5\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - 5\cdot\frac{2}{3}u^{\frac{3}{2}}\right] + C$$

$= \frac{1}{5}(x^2+5)^{\frac{5}{2}} - \frac{5}{3}(x^2+5)^{\frac{3}{2}} + C$ is fine, by me!

= Webassign May want Radical form:

$$= \frac{1}{5}\sqrt{(x^2+5)^5} - \frac{5}{3}\sqrt{(x^2+5)^3} + C$$

$$= \frac{1}{2}\left(\sqrt{x^2+5}\right)^5 - \frac{5}{3}\left(\sqrt{x^2+5}\right)^3 + C$$

$$a^{\frac{b}{c}} = \left(a^b\right)^{\frac{1}{c}} = \left(a^{\frac{1}{c}}\right)^b$$

$$= \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b = \sqrt[c]{a}^{\,5}$$

5 **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = \int_{u(a)}^{u(b)}$$

This is one way

Transition Step

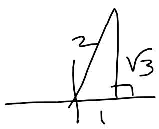
$$\int_0^{\frac{\pi}{6}} \cos(2x)\,dx = \boxed{\int_{x=0}^{x=\frac{\pi}{6}} \cos(u)\cdot\frac{1}{2}\,du} = \frac{1}{2}\int_0^{\frac{\pi}{3}} \cos(u)\,du$$

$u = 2x$

$du = 2\,dx = du$

$dx = \frac{1}{2}\,du$

$$\boxed{\begin{aligned} u &= 2x \\ u(0) &= 0 \\ u\left(\tfrac{\pi}{6}\right) &= 2\tfrac{\pi}{6} = \tfrac{\pi}{3} \end{aligned}}$$



$$= \frac{1}{2}\Big[\sin(u)\Big]_0^{\frac{\pi}{3}} = \frac{1}{2}\left[\sin\left(\frac{\pi}{3}\right) - \sin(0)\right] = \frac{1}{2}\left[\frac{\sqrt{3}}{2} - 0\right] = \frac{\sqrt{3}}{4}$$

Here's another way:

$$\int_0^{\frac{\pi}{6}} \cos(2x)\,dx = \int_{x=0}^{x=\frac{\pi}{6}} \cos(u)\cdot\frac{1}{2}\,du = \frac{1}{2}\int_{x=0}^{x=\frac{\pi}{6}} \cos(u)\,du$$

$$= \frac{1}{2}\Big[\sin(u)\Big]_{x=0}^{x=\frac{\pi}{6}} = \frac{1}{2}\sin(2x)\Big]_0^{\frac{\pi}{6}} = \frac{1}{2}\left[\sin\left(\frac{2\pi}{6}\right) - \sin(2\cdot 0)\right]$$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$, without ever re-calculating the limits of integration.

$$\int \frac{\sec^2(x)}{\tan^2(x)}\,dx = \int \frac{1}{\tan^2(x)} \cdot \sec^2(x)\,dx = \int \frac{1}{u^2} \cdot du = \int u^{-2}\,du$$

$$\sec^2(x) = \frac{d}{dx}\left[\tan(x)\right]$$

$$= \frac{u^{-1}}{-1} + C = \frac{\tan^{-1}(x)}{-1} + C$$

Let $u = \tan(x) \rightarrow$

$du = \sec^2(x)\,dx$

$$= -\tan^{-1}(x) + C$$

NOT inverse tangent, but $\frac{1}{\text{tangent}}$!

$$= -\frac{1}{\tan(x)} + C$$

$$= -\cot(x) + C\,!$$

$$\int \frac{\sec^2(x)}{\frac{\sin^2(x)}{\cos^2(x)}}\,dx = \int \sec^2(x) \cdot \frac{\cos^2(x)}{\sin^2(x)}\,dx = \int \frac{dx}{\sin^2(x)} = \int \csc^2(x)\,dx$$

$$= -\cot(x) + C\,!$$ using $\frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$!

$$\int x\,(2x+5)^8\,dx$$

$u = 2x+5$

$du = 2dx$

$dx = \frac{du}{2}$

$$= \int x\,(u)^8\,\frac{du}{2}$$

$u = 2x+5 \rightarrow$

$u-5 = 2x \Rightarrow$

$x = \frac{u-5}{2}$

$$= \int \frac{u-5}{2}\,u^8\,\frac{du}{2} = \frac{1}{4}\int u^8(u-5)\,du = \frac{1}{4}\int (u^9 - 5u^8)\,du$$

$$= \frac{1}{4}\left[\frac{u^6}{6} - 5\cdot\frac{u^9}{9}\right] + C = \frac{(2x+5)^6}{24} - \frac{5}{36}(2x+5)^9 + C$$

$$= \frac{(2x+5)^6}{24} - \frac{5(2x+5)^9}{36} + C$$

$$= \frac{1}{24}(2x+5)^6 - \frac{5}{36}(2x+5)^9 + C$$

41. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x)\, dx = 0$!

$$\underbrace{\text{ODD} + \underbrace{\text{Even} \cdot \text{ODD}}_{\text{ODD}}}_{\text{ODD}}$$

$$\int_{-a}^{a} \text{odd} = 0$$

$x^3 + x^4 \tan(x)$ is ODD!

$- + (+)(-) = - + - = -$ means ODD

$$\frac{\sin^3(x)\cos^3(x)}{\tan^4(x)+1} = \frac{(-)^3(+)^3}{(-)^4+1} = \frac{-}{+} = -\text{ is ODD!}$$