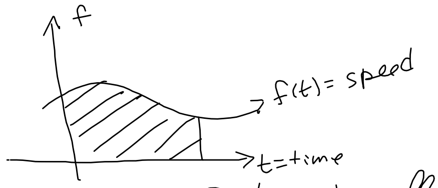
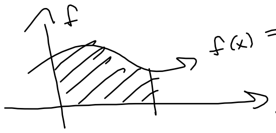
Recall
Distance = Rate · time
Work = Force · distance

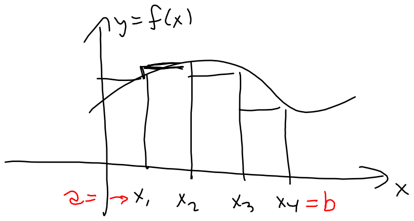


Area = Distance travelled



Area = WORK!

We formulate an estimate of area as a Riemann Sum that adds together a bunch of rectangles

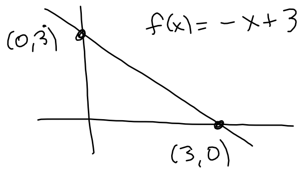


$\Delta x = \dfrac{\text{width of interval}}{\#\text{ of subintervals}} = \dfrac{b-a}{n}$

& for right endpoints / right Riemann Sums, we have $x_k = a + k\Delta x = a + k\left(\frac{b-a}{n}\right)$

$\text{Area} \approx \sum_{k=1}^{4} f(x_k)\Delta x$

In section 4.2, we say, more rectangles ⟹ better estimate, so do calculus to pass to the limit as $n \to \infty$ & find the exact area.

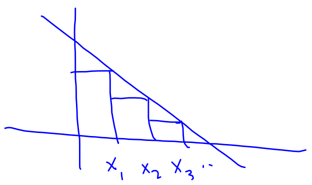


Find area under $f(x) = -x+3$ from $x=0$ to $x=3$.

$A = \frac{1}{2}bh = \frac{1}{2}\,3\cdot 3 = \frac{9}{2}$ = Exact Area.

From junior-high.

Now, with calculus idea

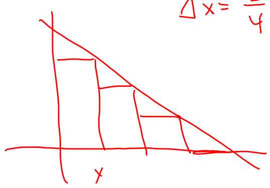


$a=0,\ b=3,\ n=n \Longrightarrow \Delta x = \frac{b-a}{n} = \frac{3}{n}$ = step length = width of all the rectangles.

$x_1 = 0 + 1\cdot\frac{3}{4} = \frac{3}{4}$, $x_2 = \frac{6}{4}$, $x_3 = \frac{9}{4}$, $x_4 = \frac{12}{4} = 3$

n=4

$\Delta x = \frac{3}{4}$



$\text{Area} \approx \sum_{k=1}^{4} f(x_k)\Delta x = \Delta x \sum_{k=1}^{4} f(x_k)$

$= \frac{3}{4}\sum_{k=1}^{4}(-x_k+3) = \frac{3}{4}\sum_{k=1}^{4}\left(-\left(\frac{3k}{4}\right)+3\right)$

$= \frac{3}{4}\left[-\frac{3}{4}+3 + -\frac{6}{4}+3 + -\frac{9}{4}+3 + -\frac{12}{4}+3\right]$

etc. Add 'em up. It's an under-estimate.

Now for Calculus!

$\sum$ is "linear"

$$5\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 5a_k = 5a_1 + 5a_2 + 5a_3 + \dots + 5a_n$$

$$\to 5(a_1 + a_2 + a_3 + \dots + a_n)$$

We proved:

$$\sum_{k=1}^{n} 7 = 7n, \quad \sum_{k=1}^{n} k = 1+2+3+\dots+n = \frac{n(n+1)}{2} = \frac{n^2 + \text{smaller}}{2}$$

Area under $f(x) = -x + 3$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$x_k = a + k\Delta x = 0 + k \cdot \frac{3}{n} = \frac{3k}{n} = x_k$$

$$\Delta x \sum_{k=1}^{n} f(x_k) = \frac{3}{n} \sum_{k=1}^{n} (-x_k + 3) = \frac{3}{n} \sum_{k=1}^{n} \left(-\frac{3k}{n} + 3\right)$$

$\sum(a_k+b_k) = \sum a_k + \sum b_k$ property. ($\sum$ is "linear")

$$= \frac{3}{n}\left(\sum_{k=1}^{n} -\frac{3k}{n} + \sum_{k=1}^{n} 3\right) \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2+\text{smaller}}{2}$$

$$= \frac{3}{n}\left(-\frac{3}{n}\sum_{k=1}^{n} k + 3n\right)$$

$$= \frac{3}{n}\left(-\frac{3}{n}\left(\frac{n^2+\sim}{2}\right) + 3n\right) = -\frac{9}{n^2}\left(\frac{n^2+\sim}{2}\right) + \frac{3}{n}\cdot 3n$$

$$= \frac{-9n^2+\sim}{2n^2} + 9 \xrightarrow{n\to\infty} \frac{-9n^2}{2n^2} + 9 = -\frac{9}{2} + 9 = \frac{9}{2}$$

Still pretty exhausting, but now we can find exact area under CURVES, like $y=x^2$!

§4.2 $\sum k = \frac{n^2+\sim}{2} = \frac{n(n+1)}{2}$

$$\sum k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+\sim}{6} = \frac{n^3+\sim}{3}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4+\sim}{4}$$

§4.3 Makes this all very SLICK! Use ANTI-Derivatives!

Fundamental Theorem of Algebra FTC I:

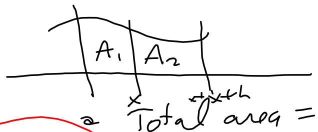
If $f(x)$ is continuous, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

FROM §4.2, we have this notion that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right).$$

Let $g(x) = \int_a^x f(t)\,dt$. Then $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$



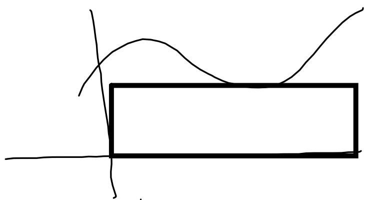
Total area $= A_1 + A_2$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \qquad \int_a^{x+h} = \int_a^x + \int_x^{x+h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(t)\,dt\right]$$

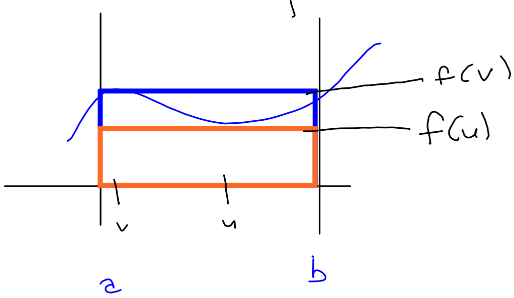
By EVT, f has a max & a min somewhere in the interval $[x, x+h]$. Let $f(u) = \min$ & $f(v) = \max$. Then

$$f(u)\,h \le \int_x^{x+h} f(t)\,dt \le f(v)\,h$$



$$f(u) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(v)$$

(assume $h>0$)



$\xrightarrow{h\to 0}$

$$f(x) \le \frac{d}{dx}\int_a^x f(t)\,dt \le f(x)$$

$\blacksquare$! Sweet!

See videos on the theory. We PROVE the following.

FTC II

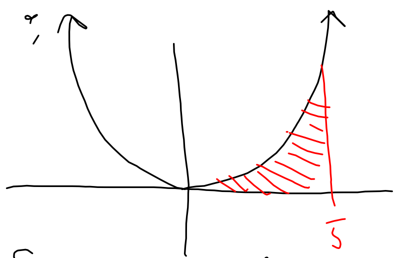
If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

$\frac{d}{dx}\left[x^3\right] = 3x^2$, so x^3 is an antiderivative of $3x^2$,

That says $\int_0^5 3x^2\,dx = \int_0^5 3t^2\,dt = x^3\Big]_0^5 = 5^3 - 0^3 = 125$

= exact area :



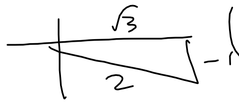
The area is SIGNED. Below x-axis ⟹ Negative Area!

What's the area under $\sec^2(x)$ from $x = -\frac{\pi}{6}$ to $\frac{\pi}{3}$?

$\int \sec^2(x)\,dx = \tan(x) + C$, so $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x)\,dx = \tan\left(\tfrac{\pi}{3}\right) - \tan\left(-\tfrac{\pi}{6}\right)$

$\frac{d}{dx}\left[\ ?\ \right] = \sec^2(x).$

$= \sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right) = \frac{3+1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$



$\boxed{= \frac{4\sqrt{3}}{3}}$

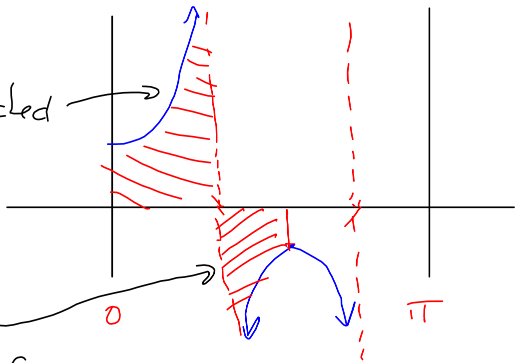
Another one w/ $\sec^2(x)$:

$$\int_0^{\pi} \sec^2(x)\,dx = \tan(\pi) - \tan(0) = 0\ ?$$

What's wrong?

Both areas are unbounded

FTC II requires continuity on [a,b]



$$\int_0^{\frac{5\pi}{6}} \sec^2(x)\,dx = \tan\left(\tfrac{5\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

Not reliable!

Always check for ~~continuity~~ DOMAIN on [a,b]!

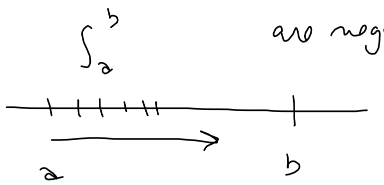
Facts About Integrals

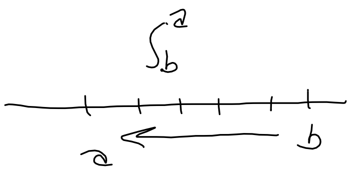
$a < c < b$

$$\int_a^b = \int_a^c + \int_c^b$$

$$\int_{x^2}^{\sin(x)} = \int_{x^2}^{0} + \int_0^{\sin(x)} = -\int_0^{x^2} + \int_0^{\sin(x)}$$

$\int_a^b = -\int_b^a$ because Now you're going Right to left & so the Δx's, ie., the dt or dx's are negative.





$$\frac{d}{dx}\int_x^5 \sin(t)\,dt = -\frac{d}{dx}\int_5^x \sin(t)\,dt = \boxed{-\sin(x)}$$

Next Level:

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\cdot u'(x)$$

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\sin(3x^2)] = \cos(3x^2)\cdot 6x \qquad \text{(Chain Rule)}$$

Pinnacle of FTC I:

Let $g(x) = \int_{\sin(x)}^{x^2+2x} \cos(t)\,dt = \int_{\sin(x)}^{0} + \int_0^{x^2+2x}$

$$= -\int_0^{\sin(x)} \cos(t)\,dt + \int_0^{x^2+2x} \cos(t)\,dt \Longrightarrow g'(x) = \frac{d}{dx}\int \sim$$

$$\boxed{= -\cos(\sin(x))\cdot\cos(x) + \cos(x^2+2x)\cdot(2x+2)}$$

§4.3 # 9.

$\boxed{FTC\ II}$

$$\int_1^9 \frac{4+x^2}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{4}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) dx \qquad \frac{x^2}{x^{1/2}} = x^{2-\frac{1}{2}}$$

$$= \int_1^9 \left(4x^{-\frac{1}{2}} + x^{\frac{3}{2}}\right) dx = \left[4\,\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_1^9$$

$$= \left[\frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_1^9 = \left[8x^{\frac{1}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_1^9 = \left(8(3) + \frac{2}{5}(3^5)\right)$$

$$-\left(8 + \frac{2}{5}\right) = 24 + \frac{2}{5}(243) = \frac{120+486}{5} = \frac{606}{5} \quad \text{if}$$

my arithmetic's right.