Today: Chapter 4 Integrals in Earnest

4.1 - Recognizing definite integrals as areas – $D = rt$ $\quad D = D_1 + D_2 + \dots + D_n = \sum_{k=1}^{n} D_n$

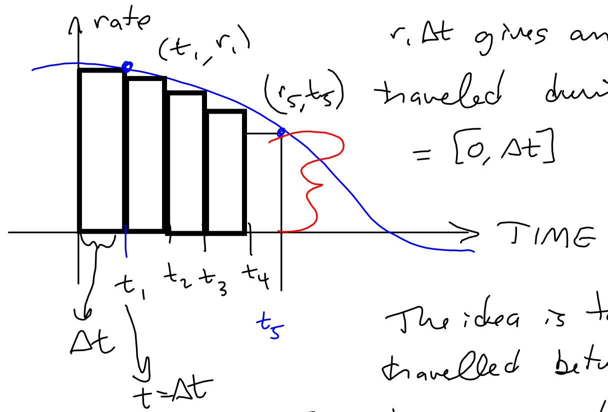
4.2 - Evaluating definite integrals by the limit definition $\quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Distance = Rate · Time

I want to convince you that the area under the rate/speed function is the DISTANCE!



$r_1 \Delta t$ gives an estimate of the distance traveled during the time period $[0, t_1]$ $= [0, \Delta t]$

The idea is to find total distance travelled between $t=0$ & $t=t_5$

It's the area under the speed curve!

Estimate the area:

$$r(t_1)\Delta t + r(t_2)\Delta t + \cdots + r(t_5)\Delta t$$

$$r_1\Delta t + r_2 \Delta t + \cdots + r_5 \Delta t$$

$$\text{Area} = \text{Distance} = \sum_{k=1}^{n} r(t_k)\Delta t$$

height width.

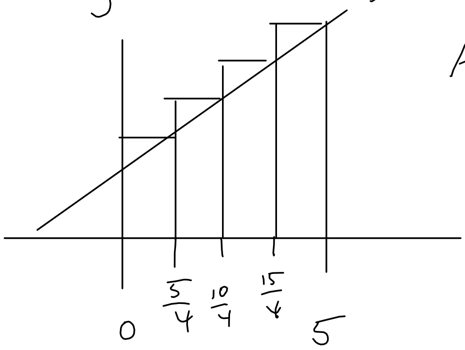
In §4.1, we're primarily using right endpoints.

This is TEDIOUS! So not many of these, especially ones like §4.1 #4.

Let's find the area under $f(x) = 3x+2$ over $[0, 5]$

estimate

using $n = 4$ rectangles & right endpoints:



$$\text{Area} \approx \sum_{k=1}^{4} f(x_k)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{4} = \frac{5}{4}$$

x_k: $x_1 = \frac{5}{4}$

$$x_2 = x_1 + \Delta x = \frac{5}{4} + \frac{5}{4} = \frac{10}{4}$$

$$x_3 = x_2 + \Delta x = \frac{10}{4} + \frac{5}{4} = \frac{15}{4}$$

$$x_4 = x_3 + \Delta x = \frac{15}{4} + \frac{5}{4} = \frac{20}{4} = 5$$

$$\text{Area} \approx \sum_{k=1}^{4} \left(3(x_k)+2\right)\cdot\frac{5}{4}$$

$$= (3x_1+2)\left(\underline{\tfrac{5}{4}}\right) + (3x_2+2)\left(\underline{\tfrac{5}{4}}\right) + (3x_3+2)\left(\underline{\tfrac{5}{4}}\right) + (3x_4+2)\left(\underline{\tfrac{5}{4}}\right)$$

$$= \frac{5}{4}\left(3x_1+2 + 3x_2+2 + 3x_3+2 + 3x_4+2\right)$$

$= \Delta x \sum_{k=1}^{4} f(x_k)$ if I'd factored out the $\frac{5}{4}$ to start with.

4.2 Theory $\sum_{k=1}^{n} c = \underbrace{c+c+c+\cdots+c}_{n \text{ of 'em}} = nc$ $\sum_{k=1}^{10} 5 = 10\cdot 5 = 50$

$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ so $\sum_{k=1}^{20} k = \frac{20(21)}{2} = 10(21) = 210$

For Integrals, think of $\frac{n(n+1)}{2} = \frac{n^2+n}{2} = \frac{n^2 + \text{something smaller}}{2}$.

$= \frac{n^2 + \sim\sim}{2}$

Principle of Mathematical Induction:

$P(n)$ is some statement that holds for $n \in \mathbb{N} = \{1,2,3,\ldots\}$

$P(n)$: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

PMI Prove $P(1)$ holds.

Assume $P(n)$ holds for some $n \in \mathbb{N}$

Prove $P(n+1)$ holds.

Conclude $P(n)$ holds for all $n \in \mathbb{N}$.

Let S = set of all $n \in \mathbb{N}$ such that $P(n)$ holds.

Show $1 \in S$ (Shows us $S \neq \emptyset$, so assuming $n \in S$ is legit)

Show if $n \in S$ then $n+1 \in S$

Conclude $S = \mathbb{N}$.

Claim: $\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$. GAUSS!

Proof Let $P(n)$ denote the statement $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Note that $P(1)$ is $\sum_{k=1}^{1} k = \frac{1(1+1)}{2} = \frac{2}{2} = 1$, so $P(1)$ holds!

Now, suppose $P(n)$ holds for some $n \geq 1$, $n \in \mathbb{N}$.

Then $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 1+2+3+\cdots+n$. Consider the sum

$$\sum_{k=1}^{n+1} k = \underbrace{1+2+3+4+\cdots+n}_{\sum_{k=1}^{n} k} + (n+1) = \frac{n(n+1)}{2} + (n+1),$$ by $P(n)$ holds.

$$= \frac{n(n+1)}{2} + (n+1)\left(\frac{2}{2}\right) = \frac{n^2+n}{2} + \frac{2n+2}{2} = \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$$, so $P(n+1)$ holds!

$\therefore$ $P(n)$ holds for all $n \in \mathbb{N}$! $\blacksquare$

Other Sum formulas?

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + \cdots}{6} = \frac{n^3 + \cdots}{3}$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{n^2+n}{2}\right]^2 = \frac{n^4 + \cdots}{4}$$

$x_k = x_{k-1} + \Delta x$

$x_1 = 0 + \Delta x = \frac{5}{4}$

$x_2 = 0 + \Delta x + \Delta x = 0 + 2\Delta x$

$x_3 = 0 + 2\Delta x + \Delta x = 0 + 3\Delta x$

$x_4 = 0 + 4\Delta x$

Right Endpoints:

$\boxed{x_k = a + k\Delta x}$

Left Endpoints:

$\boxed{x_k = a + (k-1)\Delta x}$

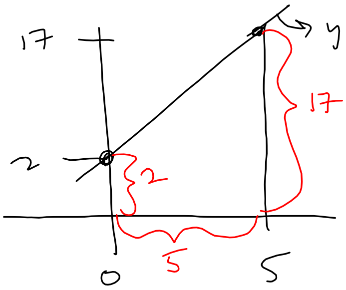
$$\sum_{k=1}^{4} f(x_k)\Delta x = \Delta x \sum_{k=1}^{4} f(x_k) =$$

$$= \frac{5}{4}\left[f\left(\frac{5}{4}\right) + f\left(\frac{10}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{20}{4}\right)\right]$$

$$= \frac{5}{4}\left[3\left(\frac{5}{4}\right) + 2 + 3\left(\frac{10}{4}\right) + 2 + 3\left(\frac{15}{4}\right) + 2 + 3\left(\frac{20}{4}\right) + 2\right]$$

$$= \frac{5}{4}\left[\frac{15}{4} + \frac{30}{4} + \frac{45}{4} + \frac{60}{4} + 8\right] = \frac{5}{4}\left[\frac{150+32}{4}\right] = \frac{5(182)}{4}$$

$$= \frac{5(91)}{2} = \boxed{\frac{455}{2} \approx \text{AREA}}$$



$3(5)+2=17$

$$\text{Trapezoid} = \tfrac{1}{2}(b_1+b_2)h$$

$$= \tfrac{1}{2}(17+2)(5) = \frac{19(5)}{2} = \frac{95}{2}$$

4

MORE RECTANGLES $\longrightarrow$ Smaller Δx $\longrightarrow$ Better Estimate

Calculus: Let $n \to \infty$!

§4.2: $n \to \infty$!

Let's Abstract the idea:

$$\Delta x = \frac{5}{4}$$

$$x_K = 0 + \frac{5}{4}K = \frac{5}{4}K$$

$$f(x_K) = 3\left(\frac{5}{4}K\right)+2$$

$$\Delta x \sum_{K=1}^{4}\left(3\left(\frac{5}{4}K\right)+2\right)$$

In General:

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

$$x_K = a + K\Delta x = 0 + K\cdot\frac{5}{n} = \frac{5K}{n}$$

$$f(x_K) = 3\left(\frac{5K}{n}\right)+2$$

$$\Delta x \sum_{K=1}^{n} f(x_K) = \frac{5}{n}\sum_{K=1}^{n}\left[3\left(\frac{5K}{n}\right)+2\right]$$

$$= \frac{5}{n}\sum_{k=1}^{n}\left(\frac{15k}{n}+2\right) = \frac{5}{n}\left[\sum_{k=1}^{n}\frac{15k}{n} + \sum_{k=1}^{n}2\right]$$

$$= \frac{5}{n}\left[\frac{15}{n}\sum_{k=1}^{n}k + 2n\right] \qquad \text{Using } \sum_{k=1}^{n}k = \frac{n(n+1)}{2} = \frac{n^2+\sim}{2}$$

$$= \frac{5}{n}\left[\frac{15}{n}\left(\frac{n^2+\cdots}{2}\right) + 2n\right]$$

$$= \frac{5}{n}\left[\frac{15n^2}{2n} + \frac{\sim}{2n} + 2n\right]$$

$$= \frac{75n^2}{2n^2} + \frac{5\sim}{2n^2} + \frac{10n}{n} = \frac{75}{2} + 10 = \frac{75+20}{2} = \frac{95}{2}$$

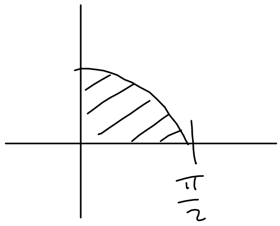
$5\sim$ is of degree 1

$$\underbrace{\int_0^5 (3x+2)\,dx = \left[\frac{3x^2}{2} + 2x\right]_0^5 = \frac{3(25)}{2} + 2(5) = \frac{95}{2}}_{\text{Checking work using FTC II from } \S 4.3.} \checkmark$$

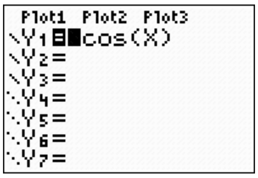
Here's a Maple Implementation. Remote section can't do this, due to access to technology issues. So here's the solution:

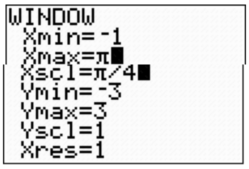
§4.1 #4

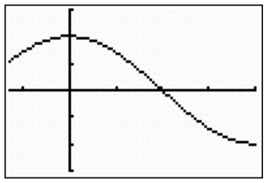
Area under $2\cos(x)$ from $x=0$ to $x=\frac{\pi}{2}$:

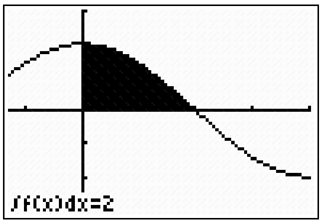


TI-84 Implementation (EXACT VALUE)









We're not set up (Remote) for you to do 10, 20, 30, ..., 100 = n Riemann Sums.

Here's Maple version:

Right Endpoints

$f(x) = 2\cos(x)$

$[a,b] = [0, \frac{\pi}{2}]$

We want:

$$\sum_{k=1}^{n} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2}}{n} = \frac{\pi}{2n}$$

$a=0$

$$x_k = a + k\Delta x = \frac{k\pi}{2n}$$

$$\text{So } \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} 2\cos\left(\frac{k\pi}{2n}\right)\cdot\frac{\pi}{2n} = \boxed{\frac{\pi}{2n}\sum_{k=1}^{n} 2\cos\left(\frac{k\pi}{2n}\right)}$$

Symbolic Setup!

$$R := n \rightarrow \frac{\text{Pi}}{2\cdot n}\cdot\sum_{k=1}^{n} 2\cdot\cos\left(\frac{k\cdot\pi}{2\cdot n}\right)$$

$$R := n \mapsto \frac{\pi\left(\sum_{k=1}^{n} 2\cos\left(\frac{k\,\pi}{2\,n}\right)\right)}{2\,n}$$

$evalf(R(10))$

1.838806341

$evalf(R(30))$

1.947183177

$evalf(R(50))$

1.968419577

$evalf(R(100))$

1.984250912

$\lim_{n\to\infty} R(n)$

2

§4.3: Fundamental Thm of Calculus II:
Find Antiderivative.
Evaluate " " at endpoints and subtract:

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x)$$

$$\int_0^{\frac{\pi}{2}} 2\cos(x)\,dx = 2\int_0^{\frac{\pi}{2}} \cos(x)\,dx = 2\sin(x)\Big]_0^{\frac{\pi}{2}} = 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0)$$

$$\boxed{= 2}$$