§2.4 $\frac{d}{dx}\sin(x)=\cos(x)$

$\frac{1}{2}$ of a proof: Assume $h>0$ & $\lim_{h\to 0}$ will be $\lim_{h\to 0^+}$ & leave $\lim_{h\to 0^-}$ as an exercise.

$f(x)=\sin x \Rightarrow$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

$$= \frac{\sin x(\cos h - 1) + \sin h\cos x}{h}$$

$$= (\sin x)\left(\frac{\cos h - 1}{h}\right) + \left(\frac{\sin h}{h}\right)\cos x \xrightarrow{h\to 0} \cos x$$

$\cos x$ is our goal

$\xrightarrow{h\to 0} 0$ we'll cop out on.

$\xrightarrow{h\to 0} 1$ will prove

Begin 2.4 discussion.

We want to prove that $\frac{d}{dx}\sin(x)=\cos(x)$

This will require a tricky limit, namely, $\lim_{h\to 0}\frac{\sin(h)}{h}=1$, which

will require the Squeeze Theorem, after convincing ourselves

that the compound inequality,

$\cos(h)<\frac{\sin(h)}{h}<1$, holds for small positive values of h.

which, when we pass to the limit as $h\to 0$ will give us

$\lim_{h\to 0}\cos(h)\le\lim_{h\to 0}\frac{\sin(h)}{h}\le 1$, i.e., what we wanted…

Finer point on this: How did $<$ become $\le$?

Consider

$f(x)=x^2$ and $g(x)=x^2+x$

Notice that for $x>0$, $g(x)$ is always bigger. But in the limit,

$\lim_{x\to 0^+} f(x)\le\lim_{x\to 0^+} g(x)$, and *not* strictly less than. So $<$ everywhere away from one point

guarantees $\le$ in the limit as you approach that point (Might be equal).

First, we'll convince ourselves that

1. $\frac{\sin(h)}{h} < 1$

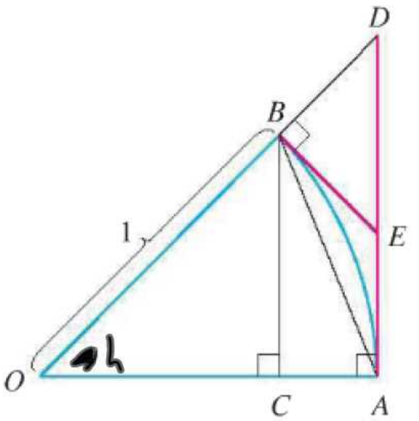
and then

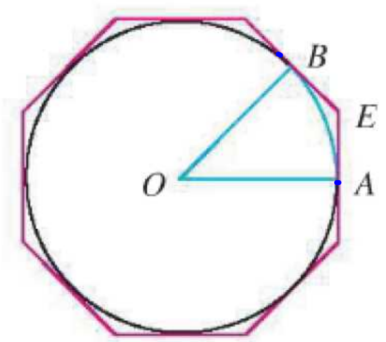
2. $\cos(h) < \frac{\sin(h)}{h}$

The desired result, $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$, will follow, immediately.

Why do we care? Because we're trying to prove the derivative of sine is cosine! And the above limit is the key.

Recall arc length $= r\theta = h$, since $r=1$, $\theta = h$ = arc AB



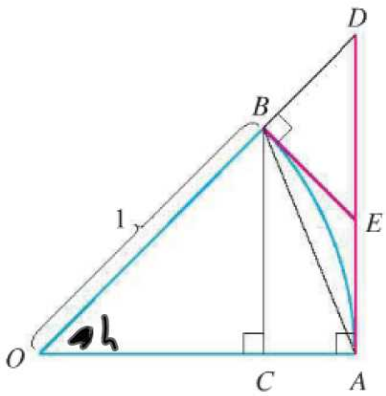


$|BC| < |AB| <$ arc $AB = h$

$\sin h < h$

$\frac{\sin h}{h} < 1$

... if we assume $h > 0$. The case $h < 0$ is an equivalent argument, so we'll let it go with the $h > 0$ case, because it's easier.



$h = \text{arc}AB < |AE| + |EB|$

$< |AE| + |ED|$

$= |AD|$, i.e.,

$h < |AD|$

Notice $\frac{|AD|}{|OA|} = \tan h$

i.e., $h < \tan h$, since $|OA| = 1$

$h < \tan h$

$h < \frac{\sin h}{\cos h} \Rightarrow$

$$\boxed{\cos h < \frac{\sin h}{h}}$$

$\frac{x+1}{x-2} > 1$

$\frac{x+1}{x-2} - 1 > 0$

$\frac{x+1-(x-2)}{x-2} > 0$

$\frac{3}{x-2} > 0$

− +

NO 2 Yes

$\{x \mid x > 2\}$

$= (2, \infty)$

Put it together:

$$\begin{array}{ccccc} \cos h & < & \dfrac{\sin h}{h} & < & 1 \\ \Big\downarrow {\scriptstyle h \to 0} & & \Big\downarrow {\scriptstyle h \to 0} & & \Big\downarrow {\scriptstyle h \to 0} \\ 1 & = & 1 & = & 1 \end{array}$$

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