MAT 201 Test 1
Spring, 2011 Name________________________
100 Points

1. The point $P=(4,2)$ lies on the graph of $f(x)=\sqrt{x}$. Let $Q=(x,\sqrt{x})$ be another point on the graph of f.

   a. (5 pts) Find the slope m_{PQ} to 4 decimal places for the following values of x:

   - $x = 3.999$

$$m_{PQ} = \frac{\sqrt{3.999}-2}{x-4} \approx .250015627 \approx \boxed{.2500}$$

   - $x = 4.001$

$$m_{PQ} = \frac{\sqrt{4.001}-2}{x-4} \approx .249984377 \approx \boxed{.2500}$$

   b. (5 pts) Based on your work for part a., estimate the slope of the tangent line $m_{\tan}$ at $x=4$.

$$\boxed{m_{\tan} \approx .2500}$$

   c. (5 pts) Based on your work for part b., construct an equation for the tangent line to $f(x)=\sqrt{x}$. (Point-slope form is just fine: $y=m(x-x_1)+y_1$.)

$$\boxed{y = .25(x-4)+2}$$

2. (10 pts) Sketch the graph of a function that meets all of the following requirements:

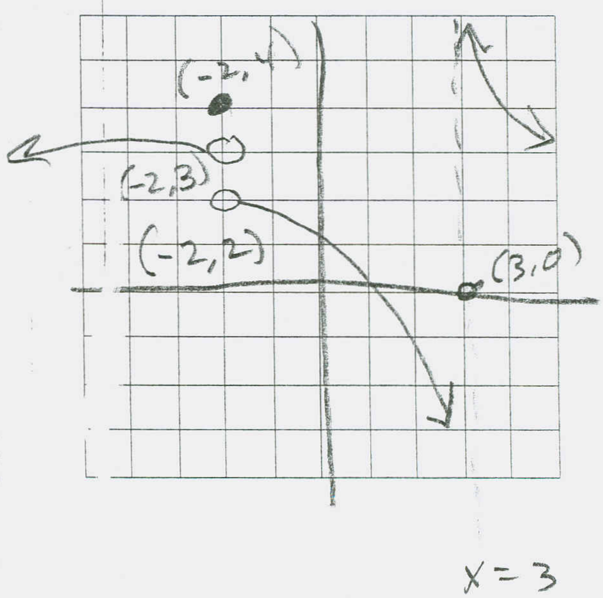
   a. $\lim_{x\to 3^-} f(x) = -\infty$

   b. $\lim_{x\to 3^+} f(x) = \infty$

   c. $\lim_{x\to -2^-} f(x) = 3$

   d. $\lim_{x\to -2^+} f(x) = 2$

   e. $f(-2)=4$



$x=3$

3. Find the limit, if it exists. If it doesn't, say so (in writing, of course).

a. (5 pts) $\lim_{x\to -5} \frac{x^2+2x-15}{x+5}$ $= \lim_{x\to -5} \frac{(x+5)(x-3)}{(x+5)} = \lim_{x\to -5}(x-3) = -8$

b. (5 pts) $\lim_{x\to 1} \frac{x^2-2x+1}{x^2+3}$ $= \frac{1^2-2(1)+1}{1^2+3} = \frac{0}{4} = 0$

c. (5 pts) $\lim_{x\to 3} \frac{x^2-2x+1}{x^2-9}$ $\not\exists$

$\to$ Not 0

$\to 0$

d. (5 pts) $\lim_{x\to -4^-} \frac{x^2-16}{|x+4|}$ $= \lim_{x\to -4^-} \frac{(x-4)(x+4)}{-(x+4)}$

$= -\lim_{x\to -4^-}(x-4) = -(-4-4) = -(-8) = 8$

20

4. (10 pts) Use the precise definition of a limit to prove that $\lim_{x\to 3}\left(\frac{1}{3}x+5\right)=6$.

Let $\varepsilon>0$ be given.

Define $\delta = 3\varepsilon$. Then

if $0<|x-3|<\delta$, we have

$\left|\frac{1}{3}x+5-6\right| = \left|\frac{1}{3}x-1\right| = \frac{1}{3}|x-3|$

$< \frac{1}{3}\delta = \frac{1}{3}\cdot 3\varepsilon = \varepsilon$ ■

Bonus 5 pts

5. ~~(5 pts)~~ Use the precise definition of a limit to prove that $\cancel{\lim_{x\to 2}(x^2-5x+7)=1}$.

$\lim_{x\to 2}(x^2+5x-6) = 8$

Scratch

$|x^2+5x-6-8|$

$= |x^2+5x-14|$

$= |x+7||x-2|$

$< |x+7|\,\delta$

Get bound

Assume $\delta \le 1$, then

$-1 < x-2 < 1$

$1 < x < 3$

$8 < x+7 < 10$

$\frac{\varepsilon}{10}$

Proof Let $\varepsilon>0$ be given. Define $\delta = \min\{1, \frac{\varepsilon}{10}\}$. Then

$0<|x-2|<\delta$ implies

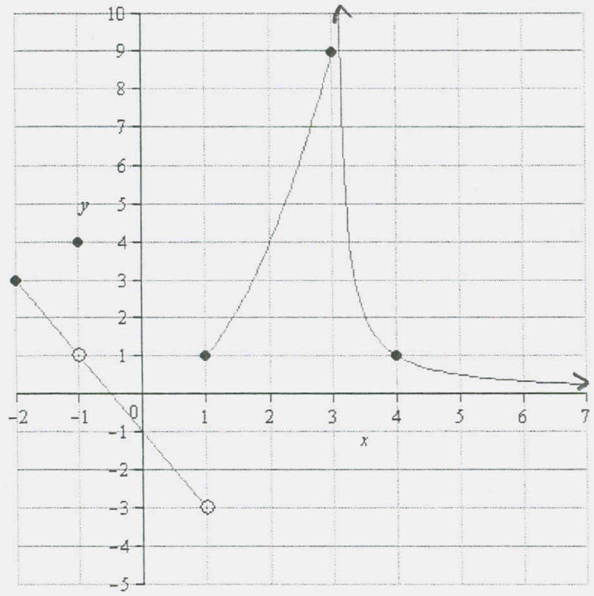
$|(x^2+5x-6)-8| = |x^2+5x-14|$

$= |x+7||x-2| < 10\delta \le 10\cdot\frac{\varepsilon}{10} = \varepsilon$

■

10

6. The graph of a function f is given below. Evaluate each limit, if it exists. If the limit does *not* exist, explain why. Employ the language (shorthand) of limits in your explanation(s), as needed.



a. (5 pts) $\lim_{x\to -1} f(x) = 1$

b. (5 pts) $\lim_{x\to 3^-} f(x) = 9$

c. (5 pts) $\lim_{x\to 3^+} f(x) = \infty$

d. (5 pts) $\lim_{x\to 3} f(x)$ $\nexists$, b/c $\lim_{x\to 3^-} f(x) = 9 \neq \infty = \lim_{x\to 3^+} f(x)$

e. (5 pts) Use the limit definition of continuity to explain why f is *not* continuous at $x = -1$.

$\lim_{x\to -1} f(x) = 1 \neq 4 = f(-1)$

f. (5 pts) Use the limit definition of continuity to explain why f *is* continuous at $x = 4$

$\boxed{\lim_{x\to 4} f(x) = 1 = f(4)}$ is why f is cnts @ $x = 4$.

20

7. Let $f(x)=\sqrt{x}$. Then the slope of the secant line between two points, $(x, f(x))$ and $(4, f(4))$, is given by the difference quotient:

$$m_{\text{sec}} = \frac{f(x)-f(4)}{x-4}.$$

a. (5 pts) Write the difference quotient for f at $x=4$. (Don't overthink this one. Just write it!)

$$\frac{\sqrt{x}-\sqrt{4}}{x-4} = \frac{\sqrt{x}-2}{x-4}$$

b. (5 pts) Based on your work in part a., compute the slope of the tangent line to f at $x=4$, by computing the limit $m_{\tan} = \lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$. Hint: Rationalize the numerator.

$$\frac{\sqrt{x}-2}{x-4} = \left(\frac{\sqrt{x}-2}{x-4}\right)\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} \quad (x\neq 4)$$

$$\xrightarrow{x \to 4} \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4} = m_{\tan}}$$

c. (5 pts) Based on your work in a. and b., construct an equation of the tangent line to f at $x=4$. If you didn't *get* b., then ***make up*** *an answer for part* b. *and use it* ! Leave your answer in point-slope form.

$$\boxed{y = \frac{1}{4}(x-4)+2}$$

15

BONUS
(5pts) Show that $x^2 = 4.1x - 2.99$ has a root in the interval $(2, 4)$ without solving the equation.

Let $f(x) = x^2 - 4.1x + 2.99$. Then

$f(2) = -1.21$ and $f(4) = 2.59$.

Since f is a polynomial, it's continuous on $[2, 4]$.

From this, and the fact that

$f(2) = -1.21 < 0$ and $f(4) = 2.59 > 0$,

the Intermediate value theorem can be applied, so $\exists\ c \in (2, 4)$ such that $f(c) = 0$. This is equivalent to saying that c is a root to the equation

$x^2 = 4.1x - 2.99$.