Miscellaneous drill on difference quotient involving square root function.
This and the next 3 pages are older stuff, re-worked a little bit after class. Might be worth a glance. Otherwise, we're starting on Page 5 of this document with new stuff.

Slope of $f(x)=\sqrt{x}$

We compute the limit as h --> 0 of the difference quotient for this function. This gives us the *slope* of the function at whatever *x* we choose to plug in (in its domain).

$(a-b)(a+b) = a^2 - b^2$

$$\frac{f(x+h)-f(x)}{h} = \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}} \xrightarrow{h\to 0} \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = f'(x)$$

More Algebra Review

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

Difference of two cubes.

Section 1.8

$\boxed{1}$ **Definition** A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

$\boxed{2}$ **Definition** A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

$\boxed{3}$ **Definition** A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

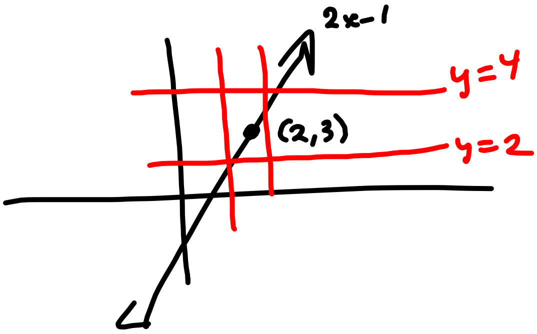
4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

I want $\lim_{x \to 2} (2x-1) = 3$

How close to 3 do you want to make

$y = 2x - 1$

within 1 unit? ok. If u insist.



Need x within $\frac{1}{2}$ unit of 2 to keep y-value within 1 unit of $y=3$.

want y within 1 unit of $y=3$

$2 < 2x-1 < 4$

$|(2x-1)-3| < 1$

$|2x-4| < 1$

$2|x-2| < 1$

$|x-2| < \frac{1}{2}$ → Then make x within $\frac{1}{2}$ unit of $x=2$

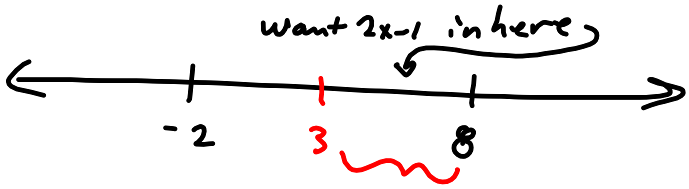
when $\varepsilon = 1$,

$\delta = \frac{1}{2}$ $\quad f(x) = 2x - 1$

Want $y=2x-1$ within 5 units of $y=3$

Want $|(2x-1)-3|<5$

want $-2 < 2x-1 < 8$



$\frac{-2+8}{2} = 3 =$ middle

$8-3=5=$ radius

$|(2x-1)-3|<5$

What condition on x guarantees this?
Reverse-engineer from result to a condition on x of the form $|x-2|<$ something.
Then we back-track.

$|(2x-1)-3|<5 \Rightarrow$

$\Leftarrow \ |2x-4|<5 \Rightarrow$

$\Leftarrow \ 2|x-2|<5 \Rightarrow$

$\Leftarrow \ |x-2|<\frac{5}{2}$

Wanted $|y-3|<5$
I got $|x-3|<\frac{5}{2}$

Yestiddy
Wante $|y-3|<1$
got $|x-3|<\frac{1}{2}$.

Want $|y-3|<\varepsilon$
Make $|x-2|<\frac{\varepsilon}{2}$

Proof

Let $\varepsilon>0$ be given.
Define $\delta=\frac{\varepsilon}{2}$. Then
any time x is within
δ units of $x=2$ $\quad (|x-2|<\delta)$
without touching $x=2$, $\quad (0<|x-2|<\delta)$

we have

$$|y-3| = |(2x-1)-3| = |2x-4|$$
$$= 2|x-2| < 2\delta = 2\left(\tfrac{\varepsilon}{2}\right) = \varepsilon \quad \blacksquare$$

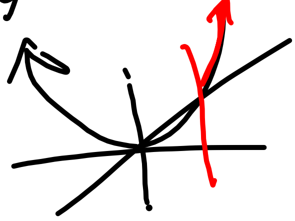
Kind of a fix from Monday's notes

§ 1.

$$\frac{x^2+1}{3x-2x^2} = \frac{x^2+1}{-x(2x-3)}$$

x-int: $x^2+1=0$ Never.

y-int: None ($\nexists$ @ $x=0$)



$\mathcal{D} = \mathbb{R} \setminus \{0, \frac{3}{2}\}$

No cancellations, so,

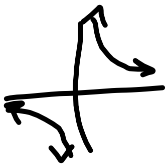
V.A. $\boxed{x=0,\ x=\frac{3}{2}}$

H.A.: $\frac{x^2}{-2x^2}$ $\boxed{\frac{1}{-2}=y}$

Power up & down is same. Look @ highest power terms.

$$\frac{x^5-7x^4+7}{7x^6-11x^5-1}$$ is proper. $\frac{\text{Degree}=5}{\text{Degree}=6}$ $\xrightarrow{x\to\infty}$ $\boxed{0=y}$

$$\frac{x^5}{7x^6} = \frac{1}{7x} = \frac{1}{7}\cdot\frac{1}{x}$$



$\frac{1}{7(1000)} \approx 0$

Also, clearly >0.

+

(x-2) positive

Different problem below this line

$$\frac{(x-2)^2(2-x)^3}{(x+1)^5(x-5)^{11}}$$

way off to the right

$$\frac{(x)^2(-x)^3}{(x)^5(x)^{11}} = \frac{(+)(-)}{(+)(+)} = -$$

−

5

Whatever else goes on to the left, we know this one is negative off to the far right.

FORMAL LIMITS. Section 1.7.

We place a requirement on how close the *y*-value must be to a specific number.

We manipulate an absolute value inequality to obtain the sufficient condition on how close *x* must be to a specific number in order to *keep* the *y*-value(s) close enough to the originally specified number.

These are *easy* to do with lines. Basically you use the reciprocal of the slope, because it is how fast the *y*-values grow, relative to the *x*-values.

It gets tricky - and forces you to *understand* - when you're up against a higher-degree polynomial. The growth rate of a polynomial is not just a constant, like you get with a straight line. But we *can* get a handle on just how bad things are (how fast *y* is growing), if we delve a little deeper into it.

Only testing over straight lines, with *BONUS* problems on higher-degree functions on the test.

APPLICATION: Kind of a 'related rates' thing. Like how fast you have to pedal in what gear to make your bicycle go 20 mph. Or how close you have to be on the *radius* of a disc in order to keep its *area* sufficiently close to a desired value.   And I think we know the formula relating area to radius... This isn't a rate but it's the same sort of deal, in practice.

Let $f(x) = x^2 - 2x - 7$

Claim $\lim_{x \to 3} f(x) = -4$

Scratch:

Want:

$|f(x) - (-4)| < \varepsilon$

$|x^2 - 2x - 7 + 4| < \varepsilon$

$|x^2 - 2x - 3| < \varepsilon$

$|x-3|\,\boxed{|x+1|} < \varepsilon$

is how close to 3 we are. Keep that. It'll be in the proof $|x-3| < \delta$

we need to know how big this might get. Let's get a handle on it by assuming x is already pretty close to x=3. Assume it's within one unit of x=3. Assume $\delta \le 1$.

So

$\delta \le 1$ means

$|x-3| \le 1$

$2 \le x \le 4$

$-1 \le x-3 \le 1$ Barely Legal.

$2 \le x \le 4$

$|x-3| \ge 1$

$-1 \ge x-3 \ge 1$ } Lies

$2+1 \le x+1 \le 4+1$

This says

$|x+1| \le 5$

Return to narrative

2 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

To prove, formally that a limit is what you say it is, you start with the very last thing in the definition:

$$|f(x) - L| < \varepsilon$$

And reason backwards to a condition placed on δ, that will guarantee the above, whenever

$$0 < |x - a| < \delta$$

It always seems to come down to algebra and analytic geometry. Got to have a handle on what things generally look like and act like.

$|x-3||x+1| < \varepsilon$

$|x-3| \cdot |x+1| \le 5|x-3|$ if we assume $\delta \le 1$

want $5|x-3| < \varepsilon$

<u>Proof</u> $|x-3| < \frac{\varepsilon}{5}$

Let $\varepsilon > 0$ be given.

Define $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then

$0 < |x-3| < \delta$ implies

$|x^2 - 2x - 7 - (-4)|$

$= |x^2 - 2x - 3| = |x-3||x+1| \le |x-3| \cdot 5$

$< \delta \cdot 5 = 5\delta \le 5 \cdot \frac{\varepsilon}{5} = \varepsilon$ ▨

Prove $\lim_{x \to 5} (x^2 - x - 14) = 6$

Scratch

$|x^2 - x - 14 - 6|$

$= |x^2 - x - 20|$ $\rightarrow$ Assume $\delta \le 1$

$= |x-5||x+4|$ Then $4 \le x \le 6$

$8 \le x+4 \le 10 \rightarrow \frac{\varepsilon}{10}$

Proof

Let $\varepsilon > 0$ be given.

Define $\delta = \min\{1, \frac{\varepsilon}{10}\}$. Then

$0 < |x-5| < \delta$ implies

$$|(x^2 - x - 14) - 6| = |x^2 - x - 20| = |x-5||x+4|$$

$$\le 10|x-5| < 10\delta \le 10 \cdot \frac{\varepsilon}{10} = \varepsilon \quad \blacksquare$$

Because delta is less than or equal to 1.

Because delta is less than or equal to epsilon/10

6 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f\big(g(x)\big) = f(b)$. In other words,

$$\lim_{x \to a} f\big(g(x)\big) = f\Big(\lim_{x \to a} g(x)\Big)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at a.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.