Section 1.6 Limit Laws: Basically they work the way you'd hope and expect. Not a lot of memorization required. Just common sense.

Limit Laws Suppose that c is a constant and the limits

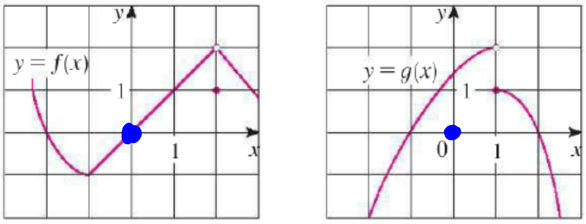
$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
    (If n is even, we assume that $a > 0$.)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer
    [If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

Only a handful, tops, of homework where you cite the specific rule being used. Otherwise, follow your instincts and seldom will you go wrong. Never ask this on a test, although failing to apply them appropriately in other context will be a problem.

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.



(a) $\lim_{x\to 2} [f(x) + g(x)]$ (b) $\lim_{x\to 1} [f(x) + g(x)]$

(c) $\lim_{x\to 0} [f(x)g(x)]$ (d) $\lim_{x\to -1} \frac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} [x^3 f(x)]$ (f) $\lim_{x\to 1} \sqrt{3 + f(x)}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)} = \frac{0}{1.3} = 0$$

$$\lim_{x\to 0} \frac{g(x)}{f(x)} \quad \nexists$$

can't play that, when limit of denominator is 0, unless limit of numerator is 0, and maybe not even then

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \quad \nexists$$

A is necessary for B

$A \Leftarrow B$

A is sufficient for B

$A \Rightarrow B$

A is necessary & sufficient for B

$A \Longleftrightarrow B$ A if and only if B

A & B are logically equivalent

$$3x + 2 = 0$$
$$\Longleftrightarrow \quad 3x = -2$$
$$\Longleftrightarrow \quad x = -\frac{2}{3}$$

$x = 5, \; x = -2$

$(x-5)(x+2) = 0$

$x^2 - 3x - 10 = 0$

Practical Limits - §1.6

2 Big Results

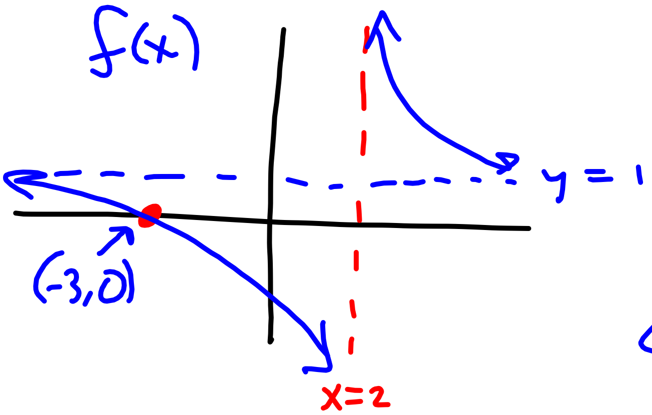
1. If $f(x) = g(x)$ everywhere, except @, say, $x=5$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ &, in particular,

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} g(x).$$

$\frac{x+3}{x-2} = f(x)$ $\qquad g(x) = \frac{x^2+2x-3}{x^2-3x+2} = \frac{(x+3)(x-1)}{(x-2)(x-1)}$

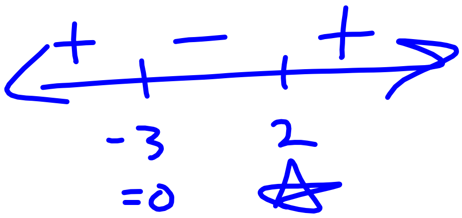
$f(x) = g(x)$ if $x \neq 1$. $\qquad = \frac{x+3}{x-2} \quad (x \neq 1)$

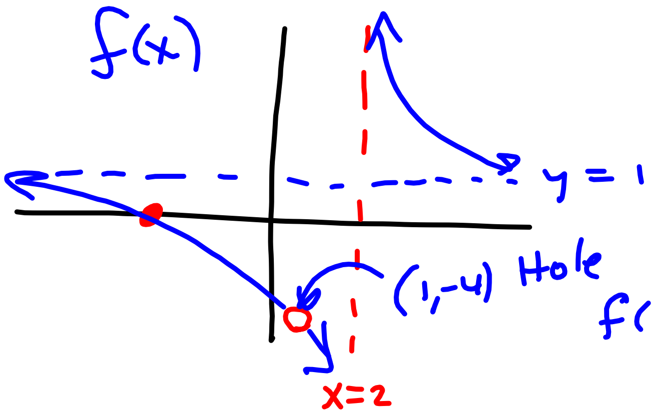
$$\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x)$$



H.A.: $\dfrac{x+3}{x-2}$

$\dfrac{x}{x} = 1 = y$





$f(1) = \dfrac{1+3}{1-2} = \dfrac{4}{-1} = -4$

$\rightsquigarrow (1,-4)$

Algebra Review

Quad Formula, Factor Theorem

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$x=c$ makes it zero $\longleftrightarrow$ $x-c$ is a factor

Factor $\overset{a}{x^2} - \overset{b}{3}x + \overset{c}{2}$

$b^2-4ac = (-3)^2 - 4(1)(2)$

$= 9 - 8 = 1$

$x = \frac{3 \pm \sqrt{1}}{2(1)} = \frac{3 \pm 1}{2}$ → 2, 1

$x=1,\ x=2$

$(x-1)(x-2)$

$(2x-3)(5x+7)$

$= 10x^2 - x - 21$

$b^2-4ac = (-1)^2 - 4(10)(-21)$

$= 1 + 840$

$= 841$

$\sqrt{841} = 29$

$x = \frac{1 \pm 29}{2(10)}$ → $\frac{30}{20} = \frac{3}{2}$, $\frac{-28}{20} = -\frac{7}{5}$

$10\left(x-\frac{3}{2}\right)\left(x+\frac{7}{5}\right)$

$2\left(x-\frac{3}{2}\right)(5)\left(x+\frac{7}{5}\right)$

$= (2x-3)(5x+7)$

3. $\lim_{t \to -2} \frac{t^4 - 2}{2t^2 - 3t + 2}$

4. $\lim_{x \to 2} \sqrt{\frac{2x^2 + 1}{3x - 2}}$

5. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

$x \neq 2$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

You should *always* hope that you can just plug in the number and get the limit at that number. This property says that you can do just that with polynomials and rational functions, assuming the rational function is defined at that number.

Of course, all the interesting limits on rational functions *won't* be defined at that number, and we'll have to do some manipulating.

Plug in if you can.
All the hard/fun ones take some manipulation

$$\lim_{x \to 2} \frac{x^2-7x+2}{x-3} = \frac{2^2-7(2)+2}{2-3}$$

lim (Poly) Always Direct Substitute

Rat'l funcs & Polys are continuous on their domains.

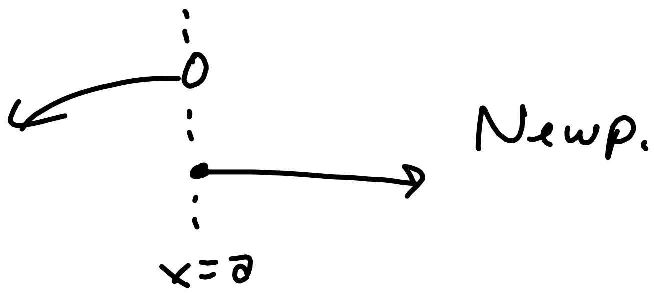
If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

You might think of this as the "guilt by association" theorem.

Alluded to this, already. For limit to exist, the left and right limits must exist and they must agree.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Non example



2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

The above is what one might call "Dominated Convergence"

If the one is above the other, everywhere *except* one spot, then, in the *limit* as you approach that spot, the limit of the one is above the limit of the other, *regardless* of what happens at the one number, itself. Limits are about what's going on *around* the point.

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Main Observation for the main application:

$$-1 \le \sin(x) \le 1 \text{ and } a \ge 0$$
$$\text{implies}$$
$$-a \le a\sin(x) \le a$$

We have some applications of this Squeeze Theorem, later. Basically, if you have a function sandwiched between two other functions, everywhere, except possibly one point, *and* if the upper and lower function have the same limit at that point, then there's no escape for the function in the middle, in the limit.

Main 'type' application:

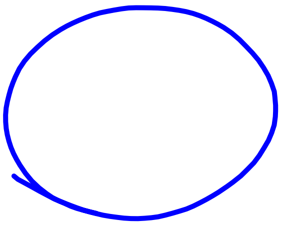
$$\lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right)$$

Damped sine curve

Dam

$x^2 \geq 0$

$$\lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right)$$



$$-1 \leq \sin\left(\tfrac{1}{x}\right) \leq 1$$

$$\begin{array}{ccccc} -x^2 & \leq & x^2 \sin\left(\tfrac{1}{x}\right) & \leq & x^2 \\ \downarrow{\scriptstyle x \to 0} & & \downarrow{\scriptstyle x \to 0} & & \downarrow{\scriptstyle x \to 0} \\ 0 & & 0 & & 0 \end{array}$$