Started out, yesterday, intending to *do* this, but somehow got sidetracked

Find slope of $f(x) = x^2 - 3x + 2$ at $x = 3$

~~This version~~ is done:

$$\frac{f(x) - f(3)}{x - 3}$$

Want to do this version:

$$\frac{f(3+h) - f(3)}{h}$$

Neither of these is ideal, since they only find the slope at one point. We want a *function* we'll call **the derivative**, that we can just plug *x*'s into and tell the slope at whatever point we choose, without having to do a difference quotient, separately, for every stinkin' one of them. I've already shown you **the derivative**, although I haven't *called* it that, just yet. But let's do the difference quotient @ *x* = 3, using the *h* --> 0 formulation, just to say we did it.

$f(x) = x^2 - 3x + 2 \implies$

$$\frac{f(x+h) - f(x)}{h} \dots$$

$$\frac{f(3+h) - f(3)}{h} = \frac{(3+h)^2 - 3(3+h) + 2 - (3^2 - 3(3) + 2)}{h}$$

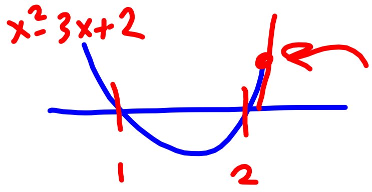
$$= \frac{9 + 6h + h^2 - 9 - 3h + 2 - (9 - 9 + 2)}{h}$$

$$= \frac{9 + 6h + h^2 - 9 - 3h + 2 - 2}{h}$$

$$= \frac{6h + h^2 - 3h}{h} = \frac{h(6 + h - 3)}{h} = 6 + h - 3 = 3 + h$$

$\dots$ $2x - 3 = f'(x)$

$\xrightarrow{h \to 0} 3 = f'(3)$ says the slope of $x^2 - 3x + 2$ @ $x = 3$.



slope @ $x = 3$ is 3.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-3(x+h)+2-(x^2-3x+2)}{h}$$

$$= \frac{x^2+2xh+h^2-3x-3h+2-x^2+3x-2}{h}$$

$f'(x)$

$$= \frac{2xh+h^2-3h}{h} = \frac{h(2x+h-3)}{h} = 2x+h-3 \xrightarrow{h\to 0} 2x-3$$

Just plug in whatever x-value to see slope.

We show the 'rationalizing the numerator' trick for limits. This comes up in 1.6 and numerous times, later. Notice in this one (as in most interesting ones), you can't just plug in x = 0, because you get 0 in the denominator. But there's hope that the limit exists, because you also get 0 in the numerator. The question is whether they're both going to zero at close enough to the same rate that the limit will exist, or not.

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

$$(a+bi)(a-bi) = a^2-(bi)^2 = a^2-b^2i^2 = a^2+b^2$$

Write $\frac{2+i}{3-i}$ in the form $a+bi$.

$$\left(\frac{2+i}{3-i}\right)\left(\frac{3+i}{3+i}\right) = \frac{6+2i+3i+i^2}{3^2+1^2} = \frac{6+5i-1}{10} = \frac{5+5i}{10} = \frac{1}{2}+\frac{1}{2}i$$

Rationalizing "thingie."

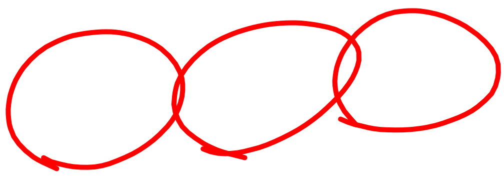
$$(a-b)(a+b) = a^2-b^2$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{1}{2+\sqrt{3}} = \left(\frac{1}{2+\sqrt{3}}\right)\left(\frac{2-\sqrt{3}}{2-\sqrt{3}}\right) = \frac{2-\sqrt{3}}{2^2-\sqrt{3}^2} = \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$$

Hard $1.41421\overline{)1.00000}$

Easy $2\overline{)1.41421}$



Rationalize NUMERATOR

$\frac{\sqrt{x^2+9}-3}{x^2}$ is $\frac{0}{0}$ situation @ $x=0$

$$\left(\frac{\sqrt{x^2+9}-3}{x^2}\right)\left(\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}\right)=\frac{x^2+9-9}{x^2\left(\sqrt{x^2+9}+3\right)}$$

$$=\frac{x^2}{x^2\left(\sqrt{x^2+9}+3\right)}=\frac{1}{\sqrt{x^2+9}+3}\xrightarrow{x\to 0}\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

Difference quotient for $f(x)=\sqrt{x}$

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$=\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}\xrightarrow{h\to 0}\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

Finish **S 1.5** with infinite limits and a quick demonstration of the graph of a rational function, which entering calculus students usually have forgotten.

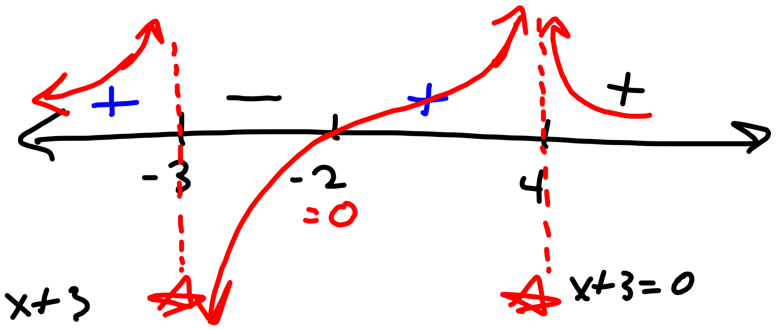
We begin with an elegant approach to **sign patterns**. If you can't do sign patterns quickly and efficiently, then Calculus will kick your butt. Most students need reminding and most students eventually get pretty good at it.

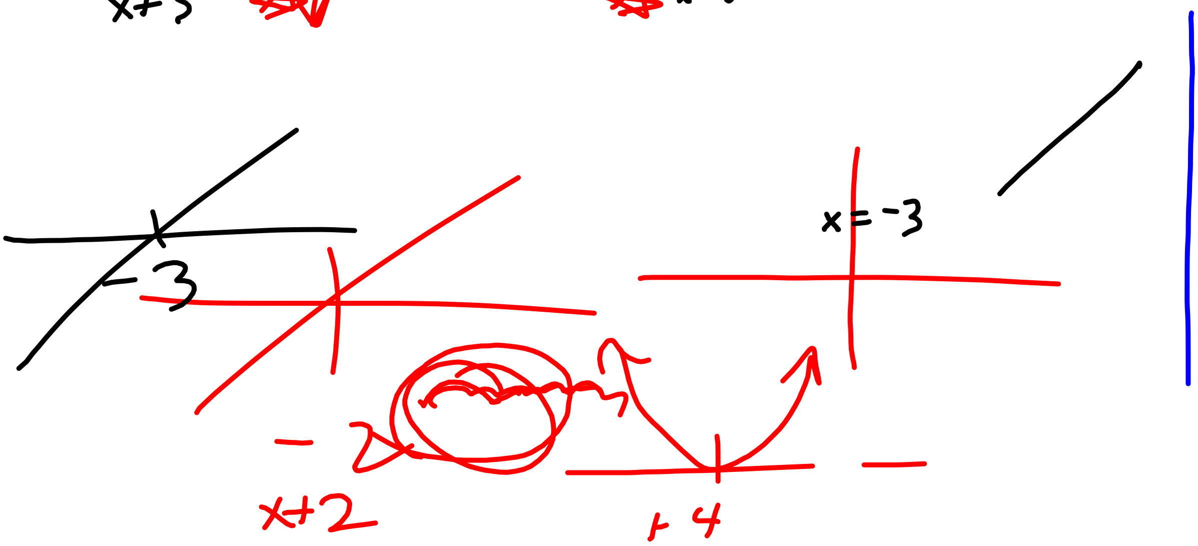
$\lim_{x\to -3^-} f(x)=\infty$, $\lim_{x\to -3^+} f(x)=-\infty$, $\lim_{x\to -3} f(x) \nexists$ $\qquad \frac{+}{(+)(+)^2}$

$\lim_{x\to 4} f(x)=\infty$, where $f(x) = \frac{(x+2)}{(x+3)(x-4)^2}$

Sign Pattern for $f(x)$

Quick, dirty sketch.

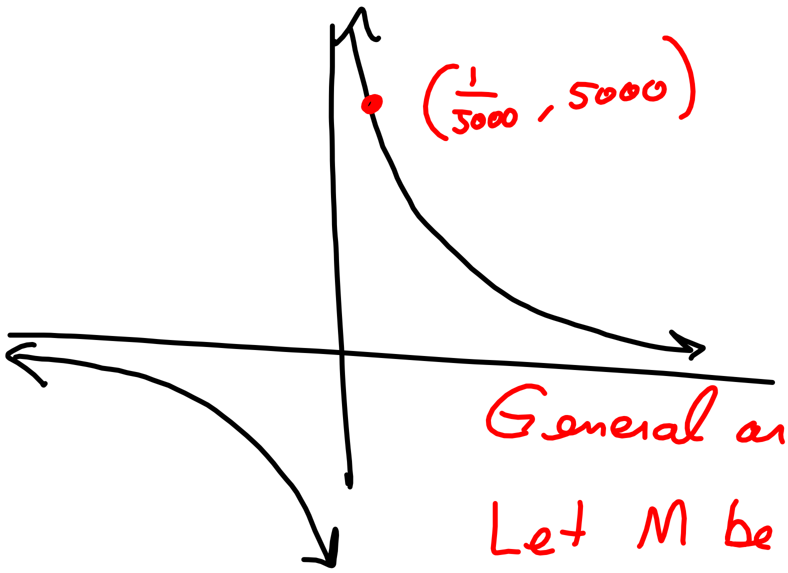




$$\lim_{x \to 2} f(x) = \infty$$

Challenge: Make $f(x)$ bigger than some big number $5{,}000 = M$

I can make $\frac{1}{x}$ bigger than 5000 by taking x less than $\frac{1}{5000}$



$f\left(\frac{1}{5000}\right) = \frac{1}{\frac{1}{5000}}$

$= 5000$

General argument!

Let M be given

Want: $\frac{1}{x} > M$

Assume $M > 0$, Assume $x > 0$ Then

$1 > Mx$

$\boxed{\frac{1}{M} > x}$

Section 1.6 Limit Laws: Basically they work the way you'd hope and expect. Not a lot of memorization required. Just common sense.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
    (If n is even, we assume that $a > 0$.)
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer
    $\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Only a handful, tops, of homework where you cite the specific rule being used. Otherwise, follow your instincts and seldom will you go wrong. Never ask this on a test, although failing to apply them appropriately in other context will be a problem.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

You should *always* hope that you can just plug in the number and get the limit at that number. This property says that you can do just that with polynomials and rational functions, assuming the rational function is defined at that number.

Of course, all the interesting limits on rational functions *won't* be defined at that number, and we'll have to do some manipulating.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

You might think of this as the "guilt by association" theorem.

Alluded to this, already. For limit to exist, the left and right limits must exist and they must agree.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

The following is what one might call "Dominated Convergence"

If the one is above the other, everywhere *except* one spot, then, in the *limit* as you approach that spot, the limit of the one is above the limit of the other, *regardless* of what happens at the one number, itself. Limits are about what's going on *around* the point.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

We have some applications of this Squeeze Theorem, later. Basically, if you have a function sandwiched between two other functions, everywhere, except possibly one point, *and* if the upper and lower function have the same limit at that point, then there's no escape for the function in the middle, in the limit.

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

Main Observation for the main application:

$$-1 \le \sin(x) \le 1 \text{ and } a \ge 0$$
implies
$$-a \le a\sin(x) \le a$$

Main 'type' application:

$$\lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right)$$