MAT 201, Test 4, Spring, 2011
100 Points

Name KEY

1. (20 pts) Evaluate the definite integral $\int_0^1 \left(6x^5 - 6x^2 + 1\right)dx$

$$= \left[x^6 - 2x^3 + x \right]_0^1 = 1 - 2 + 1 = 0$$

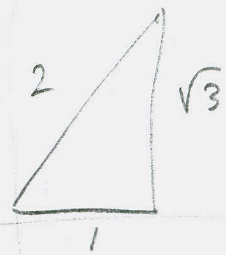
2. (10 pts) Evaluate the definite integral $\int_0^{\frac{\pi}{6}} (\cos(2t))dt$

$$\left(\begin{array}{l} u = 2t,\ du = 2dt \Rightarrow dt = \frac{du}{2} \\ t = 0 \Rightarrow u = 0,\quad t = \frac{\pi}{6} \Rightarrow u = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3} \end{array} \right)$$

$$= \int_0^{\frac{\pi}{3}} \cos(u) \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\frac{\pi}{3}} \cos(u)\,du = \frac{1}{2}\left[\sin(u)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\sin\left(\frac{\pi}{3}\right) - \sin(0)\right] = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{3}}{4}}$$

$\approx .4330127019$



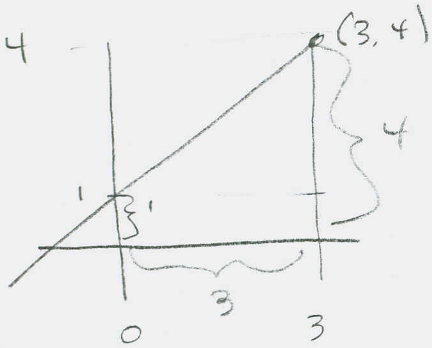
3. (10 pts) Evaluate the indefinite integral $\int \left(4x^3 - 6x^2\right)\sqrt[3]{x^4 - 2x^3 + 1}\,dx$.

$$\left(u = x^4 - 2x^3 + 1 \Rightarrow du = (4x^3 - 6x^2)dx \Rightarrow dx = \frac{du}{4x^3 - 6x^2} \right)$$

$$= \int (4x^3 - 6x^2)(x^4 - 2x^3 + 1)^{\frac{1}{3}} \left(\frac{du}{4x^3 - 6x^2}\right) = \int u^{\frac{1}{3}} du = \frac{3}{4}u^{\frac{4}{3}} + C$$

$$= \boxed{\frac{3}{4}\left(x^4 - 2x^3 + 1\right)^{\frac{4}{3}} + C}$$

4. (10 points) Evaluate $\int_0^3 (1+x)dx$ by interpreting it in terms of areas of one or more common geometric figures. In other words: Draw the picture! Find the area! You may use integration to *check* your answer, but I want to see that you know the picture and can figure the area by multiple means.



$A = \frac{1}{2}(B_1 + B_2)h$

$= \frac{1}{2}(1+4)(3) = \boxed{\frac{15}{2}} = 7.5$

Check: $\int_0^3 (1+x)\,dx = \left[x + \frac{1}{2}x^2\right]_0^3 = 3 + \frac{9}{2} = \frac{15}{2}$ ✓

5. (10 pts) Find $g'(x)$, if $g(x) = \int_0^{2x^3} \frac{\sin(\pi t)}{t^2+1}\,dt$, i.e., evaluate $\frac{d}{dx}\left[\int_0^{2x^3} \frac{\sin(\pi t)}{t^2+1}\,dt\right]$.

$= \left(\frac{\sin(2\pi x^3)}{4x^6 + 1}\right)(6x)$

6. (5 pts) Expand $\left(5 + \frac{4k}{n}\right)^3 = 5^3 + 3(5)^2\left(\frac{4k}{n}\right) + 3(5)\left(\frac{4k}{n}\right)^2 + \left(\frac{4k}{n}\right)^3$

$= \boxed{125 + 300\frac{k}{n} + 240\frac{k^2}{n^2} + 64\frac{k^3}{n^3}}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(Bonus 5 pts) Expand $(x-2)^5$.

$= (x)^5 + 5(x)^4(-2) + 10(x)^3(-2)^2 + 10(x)^2(-2)^3$

$+ 5(x)(-2)^4 + (-2)^5$

$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

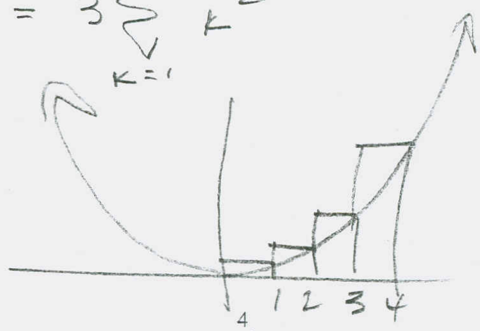
7. a. (10 pts) Estimate the area under $f(x) = 3x^2$ on the interval $[0,4]$, using a Riemann sum, with 4 subintervals, and right endpoints. Supply a sketch that shows the graph of the function and the rectangles you use in your Riemann sum.

$n=4$, $[a,b]=[0,4]$, $\frac{b-a}{n} = 1 = \Delta x$, $x_k = a + k\Delta x = k$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{4} 3x_k^2 \cdot 1 = 3\sum_{k=1}^{4} x_k^2 = 3\sum_{k=1}^{4} k^2$$

$$= 3[1^2+2^2+3^2+4^2]$$

$$= 3[1+4+9+16] = \boxed{90}$$



b. (10 pts) Use the definition of the definite integral to evaluate $\int_0^4 3x^2\,dx$. This is the *long way*. Hint: $\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n^3 + \text{lower degree}}{3}$.

$\Delta x = \frac{4}{n}$

$x_k = a + k\Delta x = 0 + k\cdot\frac{4}{n} = \frac{4k}{n}$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} 3x_k^2 \cdot \frac{4}{n} = \sum_{k=1}^{n} 3\left(\frac{4k}{n}\right)^2\cdot\frac{4}{n} = \sum_{k=1}^{n} \frac{3\cdot 16\cdot 4k^2}{n^3}$$

$$= \frac{3(64)}{n^3}\sum_{k=1}^{n} k^2 = \frac{3(64)}{n^3}\left[\frac{n^3 + \text{smaller degree}}{3}\right]$$

$$= 64\left[\frac{n^3 + \text{lesser degree}}{n^3}\right] \xrightarrow{n\to\infty} \boxed{64}$$

8. (10 pts) **Two versions of the same question:** The mountain trail from Ken's cabin takes a bee-line to Barbie's cabin over rough terrain. As the crow flies, the distance between the cabins is 4 miles. The steepness of the trail, measured in feet per mile, is described by the function $f(x) = 3x^2 - 6x + 2$. Whose cabin sits higher and by how much?

The slope (in feet per mile) of a trail is described by the function $f(x) = 3x^2 - 6x + 2$, where x measures the horizontal distance (map distance) from the trail head. What's the net change in altitude for a mountaineer who covers 4 miles (map distance) from the trail head?

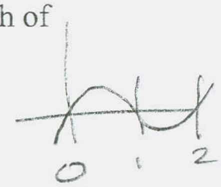
$$\int_0^4 (3x^2-6x+2)\,dx = \left[x^3 - 3x^2 + 2x\right]_0^4 = 64-48+8 = \boxed{24 \text{ ft.}}$$

Barbie's house is 24 ft above Ken's.

Net change is 24 ft.

(Bonus 5 pts) Evaluate $\int_1^3 \left|x^3 - 3x^2 + 2x\right| dx$. Accompany your work with a sketch of the area(s) involved.

$$x(x^2-3x+2) = x(x-2)(x-1)$$



$$= \cancel{\int_0^1 (x^3-3x^2+2x)\,dx} - \int_1^2 (x^3-3x^2+2x)\,dx + \int_2^3 (x^3-3x^2+2x)\,dx$$

$$= \cancel{\left[\tfrac14 x^4 - x^3 + x^2\right]_0^1} - \left[\tfrac14 x^4 - x^3 + x^2\right]_1^2 + \left[\tfrac14 x^4 - x^3 + x^2\right]_2^3$$

$$= \cancel{\tfrac14} - 1 + 1 - \left[\left(\tfrac14(16) - 8 + 4\right) - \left(\tfrac14 - 1 + 1\right)\right] + \left[\left(\tfrac14(81) - 27 + 9\right) - \left(\tfrac14(16) - 8 + 4\right)\right]$$

$$= \cancel{\tfrac14} - \left[4 - 4 - \tfrac14\right] + \left[\tfrac{81}{4} - 18 - (4-8+4)\right] = \boxed{\cancel{\tfrac14}} + \tfrac14 + \tfrac{81}{4} - 18$$

$$= \cancel{\tfrac{83}{4}} - \tfrac{72}{4} = \boxed{\cancel{\tfrac{11}{4}}} = \cancel{2.75}$$

82/4

10/4

10/4 = 2.5