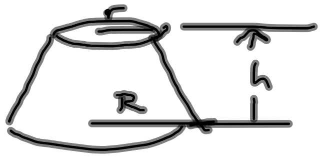
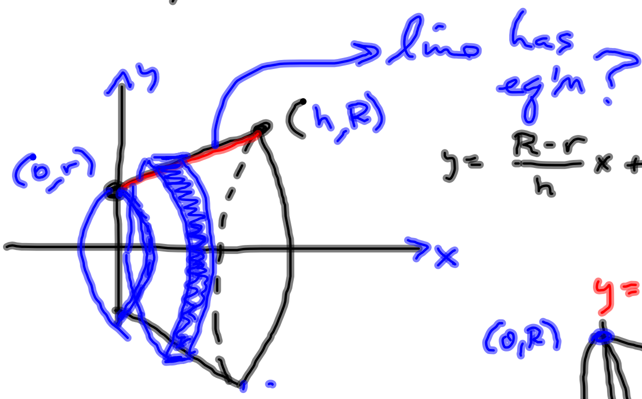
§6.2 #50 Frustum

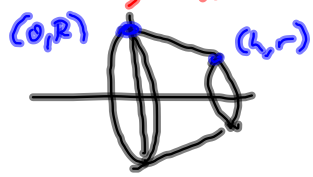
I advise MORE §5.5 practice

Find Vol of





$y = \frac{R-r}{h}x + r$



$y = \frac{r-R}{h}x + R$

$\frac{r-R}{h-0}$

$$\pi\int_0^h\left(\frac{R-r}{h}x+r\right)^2dx$$

$$= \pi\int_0^h\left(\frac{R^2-2Rr+r^2}{h^2}x^2+2r\left(\frac{R-r}{h}\right)x+r^2\right)dx$$

$(a+b)^2 = a^2+2ab+b^2$

$(a-b)^2 = a^2-2ab+b^2$

$\left(\frac{R-r}{h}x\right)^2 + 2\left(\frac{R-r}{h}x\right)r + r^2$

$= \boxed{\left(\frac{R-r}{h}\right)^2}x^2 + 2r\left(\frac{R-r}{h}\right)x + r^2$

$= \frac{R^2-2Rr+r^2}{h^2}x^2 + 2\left(\frac{R-r}{h}\right)x + r^2$

$$= \pi\left(\frac{R^2-2Rr+r^2}{h^2}\right)\int_0^h x^2dx$$

$$+\pi\cdot 2r\frac{R-r}{h}\Big)\int_0^h x\,dx$$

$$+\pi r^2\int_0^h dx$$

$$= \pi\left(\frac{R-r}{h}\right)^2\left[\frac{1}{3}x^3\right]_0^h$$

$$+\frac{2\pi r(R-r)}{h}\left[\frac{1}{2}x^2\right]_0^h$$

$$+\pi r^2\left[x\right]_0^h$$

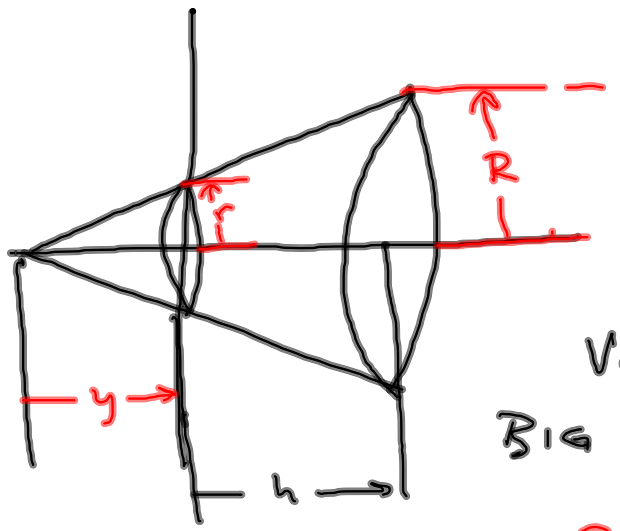
$$= \pi\left(\frac{R-r}{h}\right)^2\left[\frac{1}{3}h^3\right] + \frac{2\pi r(R-r)}{h}\left[\frac{1}{2}h^2\right] + \pi r^2 h$$

$$= \pi\left[\frac{R^2-2Rr+r^2}{h^2}\cdot\frac{h^3}{3} + \frac{2rR-2r^2}{h}\cdot\frac{h^2}{2} + r^2h\right]$$

$$= \pi\left[\frac{(R^2-2Rr+r^2)h}{3} + \frac{(\cancel{2}rR-\cancel{2}r^2)h}{\cancel{2}} + r^2h\right]$$

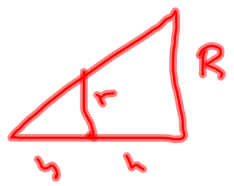
$$= \pi h\left[\frac{1}{3}R^2 - \frac{2}{3}Rr + \frac{1}{3}r^2 + rR - r^2 + r^2\right]$$

$$= \pi h\left[\frac{1}{3}R^2 + \frac{1}{3}Rr + \frac{1}{3}r^2\right] = \boxed{\frac{\pi h}{3}\left[R^2+Rr+r^2\right]}$$



Volume of frustum =
Big Cone - Little cone

Similar triangles : $\frac{R}{h+y} = \frac{r}{y}$



$V = \frac{1}{3}\pi r^2 h$

BIG - LITTLE

$= \frac{1}{3}\pi \left[R^2(h+y) - r^2 y\right]$

$\frac{R}{h+y} = \frac{r}{y} \Longrightarrow Ry = r(h+y) = rh + ry$

$Ry - ry = rh$

$$y(R-r) = rh$$

$$y = \frac{rh}{R-r} \Longrightarrow$$

$$V = \frac{1}{3}\pi\left[R^2\left(h+\frac{rh}{R-r}\right) - r^2\left(\frac{rh}{R-r}\right)\right]$$

$$= \frac{1}{3}\pi\left[R^2h + \frac{R^2rh}{R-r} - \frac{r^3h}{R-r}\right] =$$

$$= \frac{1}{3}\pi\left[\frac{R^2h(R-r) + R^2rh - r^3h}{R-r}\right]$$

$$= \frac{\pi h}{3}\left[\frac{R^3 - R^2r + R^2r - r^3}{R-r}\right] = \frac{\pi h}{3}\left[\frac{R^3-r^3}{R-r}\right]$$

$$= \frac{\pi h}{3}\left[\frac{(R-r)(R^2+Rr+r^2)}{R-r}\right] = \frac{\pi h}{3}\left[R^2+Rr+r^2\right]$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

Let $f(x)$ be continuous on $[a,b]$.

Then the average value of $f(x)$ on $[a,b]$

is $\frac{1}{b-a}\int_a^b f(x)\,dx$

$$\sum f(x)\Delta x = \sum f(x)\frac{b-a}{n} = \sum \frac{f(x)}{n}(b-a)$$

Average of n $f(x)$ values.

So $\underbrace{\frac{1}{b-a}\sum \frac{f(x)}{n}(b-a)}$ $= \sum \frac{f(x)}{n}$ is the average.

$n \to \infty$

$\frac{1}{b-a}\int_a^b f(x)\,dx$

Let $f(x)$ be continuous on $[a,b]$.

Then the average value of $f(x)$ on $[a,b]$.

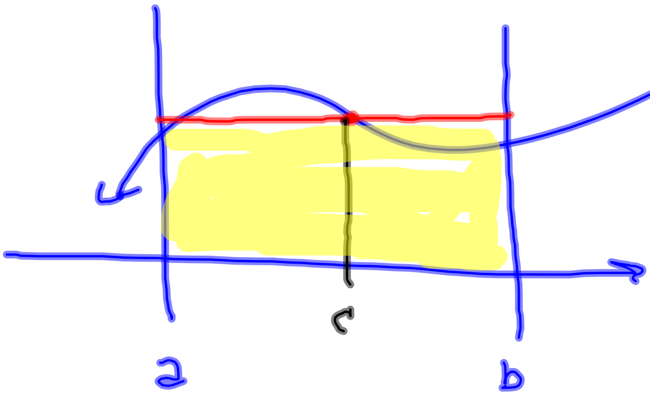
MVT Says:

There is a $c \in (a,b)$ such that

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

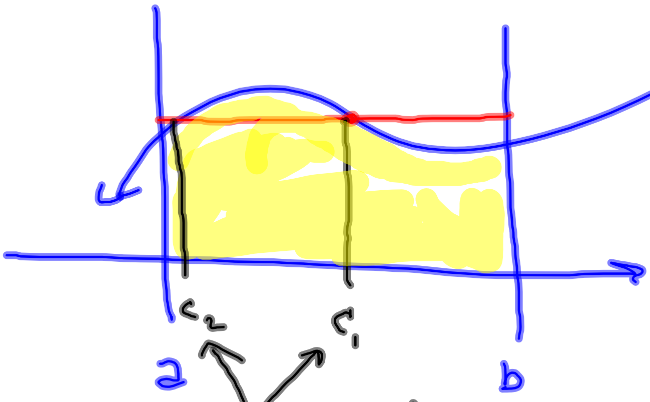
The function achieves its average value at least once in (a,b).



The area inside the rectangle is

$f(c)(b-a)$

is the same as the area under $f(x)$



Either one could be "our" c.

The area inside the rectangle is

$f(c)(b-a)$

is the same as the area under $f(x)$

Find the c satisfying the conclusion of the MVT for $f(x) = 3x^2 - 2x + 5$ on $[0,3]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(c) = \frac{1}{3}\int_0^3 (3x^2 - 2x + 5)\,dx$$

$$= \frac{1}{3}\left[x^3 - x^2 + 5x\right]_0^3$$

$$= \frac{1}{3}\left[27 - 9 + 15\right] = \frac{1}{3}[33] = 11$$

$f(c) = 11$

$$3c^2 - 2c + 5 = 11$$

$$3c^2 - 2c - 6 = 0$$

$a = 3, b = -2,$ "c" $= -6$

$$b^2 - 4ac = (-2)^2 - 4(3)(-6)$$

$$= 4 + 72$$

$$= 76$$

$$\begin{array}{r|l} 2 & 76 \\ 2 & 38 \\ & 19 \end{array}$$

So,

$$\sqrt{76}$$

$$= \sqrt{2 \cdot 2 \cdot 19}$$

$$= 2\sqrt{19}$$

$$x = \frac{2 \pm \sqrt{76}}{2(3)} = \frac{2 \pm 2\sqrt{19}}{2(3)}$$

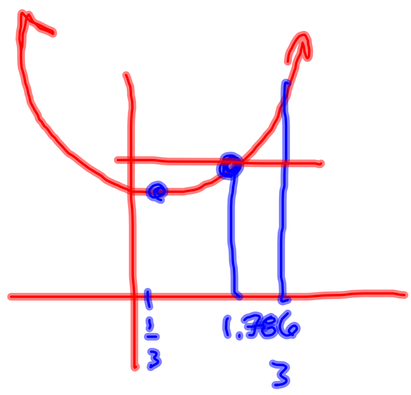
$$= \frac{1 \pm 1\sqrt{19}}{3} = \frac{1 \pm \sqrt{19}}{3} \longrightarrow \boxed{c = \frac{1 + \sqrt{19}}{3}}$$

≈ 1.786

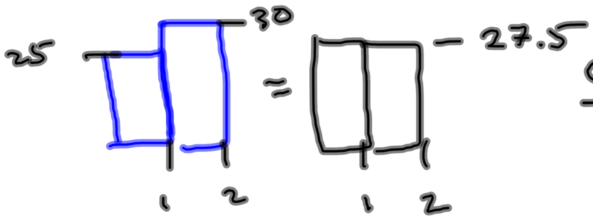
$\frac{1-\sqrt{19}}{3} \approx -1.120$

$f(x) = 3x^2 - 2x + 5$

$-\frac{b}{2a} = \frac{1}{3}$



$\frac{25+30}{2} = 27.5$



Same areas

$-x^2+6x-8 = y$

$x^2-6x+8 = -y$

$x^2-6x = -y-8$

$x^2-6x+3^2 = -y-8+9$

$\boxed{(x-3)^2 = 1-y}$

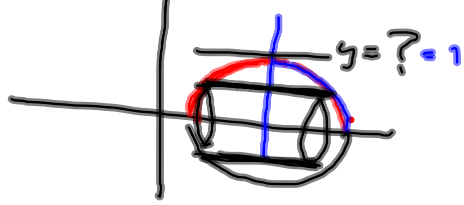
$x-3 = \pm\sqrt{1-y}$

$x = 3\pm\sqrt{1-y}$

$1-y = (x-3)^2$

$-y = (x-3)^2-1$

$y = -(x-3)^2+1$



$x = 3+\sqrt{1-y}$

$x = 3-\sqrt{1-y}$

height of cylinders =

$3+\sqrt{1-y}-(3-\sqrt{1-y})$

$= 2\sqrt{1-y}$

$2\pi\int_0^1 2\sqrt{1-y}\,dy$

$2\pi r h\Delta y$

$2\pi y(2\sqrt{1-y})\,dy$

$\boxed{=\left(2\int_0^1 y(1-y)^{\frac{1}{2}}dy\right)(2\pi)}$

$u = 1-y$

$du = -dy$

$y = 0 \Rightarrow u = 1$

$y = 1 \Rightarrow u = 0$

$= -2\pi\int_0^1 (1-y)^{\frac{1}{2}}(-dy)$

$= -2\pi\int_1^0 u^{\frac{1}{2}}du$

$= \pi 2\int_0^1 u^{\frac{1}{2}}du = \frac{4}{3}u^{\frac{3}{2}}\Big]_0^1 = \frac{4}{3}$

$$\boxed{= \left(2\int_0^1 y(1-y)^{\frac{1}{2}}\,dy\right)(2\pi)}$$

$$= 4\pi \int_0^1 y\,(1-y)^{\frac{1}{2}}\,dy$$

$u = 1-y \Rightarrow u-1=-y \Rightarrow$

$du = -dy$ $\qquad y = 1-u$

$y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=0$

$$= -4\pi \int_0^1 y\,(1-y)^{\frac{1}{2}}(-dy)$$

$$= -4\pi \int_1^0 (1-u)\,u^{\frac{1}{2}}\,du = 4\pi \int_0^1 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$