§4.3 #s 6, 18, 32, 56b
3 3 3 2

(6) Increasing: $(0,1) \cup (3,5)$
Decreasing: $(1,3) \cup (5,6)$

membership $x \in (1,3)$ OR $(5,6)$

(18) $f'(x) = 4x^3(x-1)^3 + x^4(3(x-1)^2)$

$= x^3(x-1)^2\left[\frac{4x^3(x-1)^3}{x^3(x-1)^2} + \cdots\right]$

$= x^3(x-1)^2[4(x-1) + 3x]$

$= x^3(x-1)^2[7x-4] \overset{\text{set}}{=} 0 \Rightarrow$

$x = 0, 1, 4/7$

$(fgh)' = ((fg)h)' = (fg)'h + (fg)h'$

$= f'gh + fg'h + fgh'$

$f''(x) = 3x^2[(x-1)^2(7x-4)] + 2(x-1)[x^3(7x-4)]$

$+ 7[x^3(x-1)^2]$ * Could safely do same thing with this form.

$= x^2(x-1)[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)]$

$f''(0) = 0$

$f''(1) = 0$

$f''(\frac{4}{7}) = \underbrace{(\frac{4}{7})^2(\frac{4}{7}-1)}_{<0}[0 + 0 + \underbrace{7(\frac{4}{7})(\frac{4}{7}-1)}_{<0}]$

$f''(\frac{4}{7}) > 0$

$f(\frac{4}{7})$ is a min by 2^{nd} Deriv. Test $\cup$

(c) $f(0), f(1)$ the test is inconclusive.

$f'(x_1) = 0$ Flat spot

$f''(x_1) > 0$ $\cup$ min

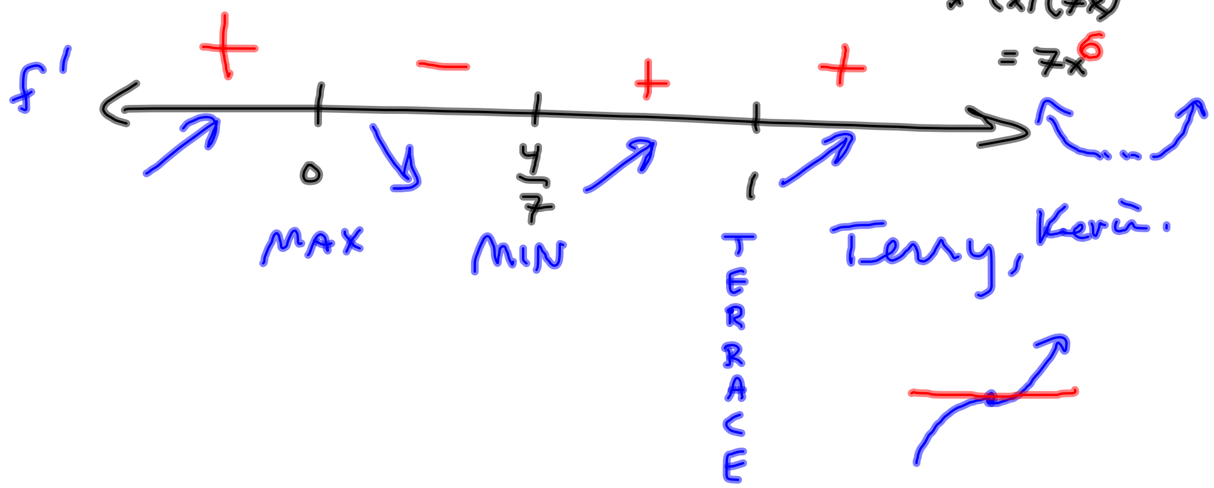
$f''(x_1) < 0$ $\cap$ Max

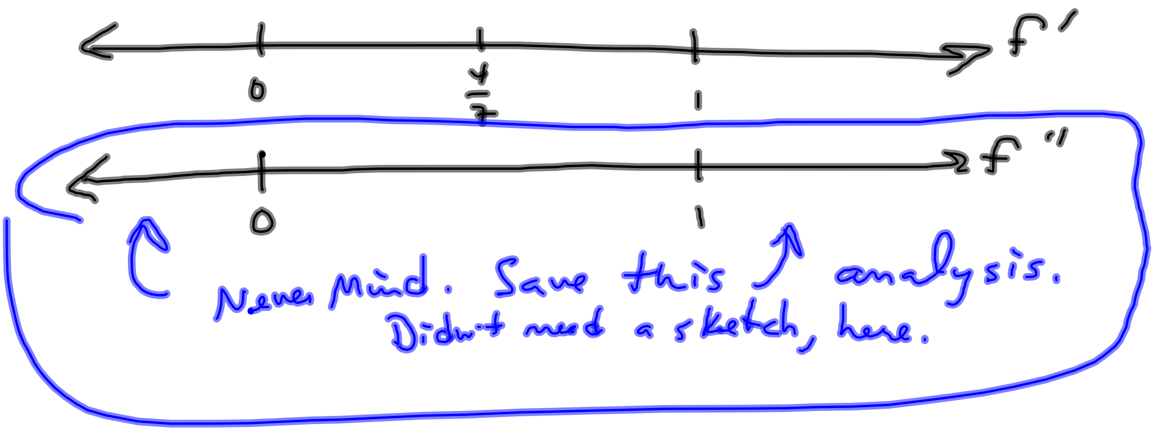
$x^3(x-1)^2(7x-4) = f'(x)$

E.B.:

$x^3(x)^2(7x)$

$= 7x^6$





32

$g(x) = x^4 + 8x^3 + 20$

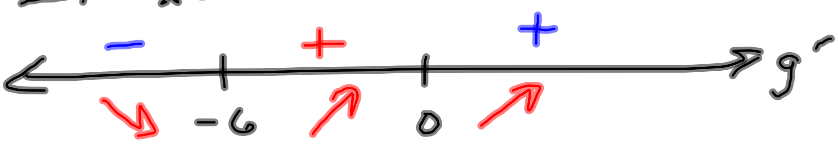
$g'(x) = 4x^3 + 24x^2 \overset{\text{SET}}{=} 0$

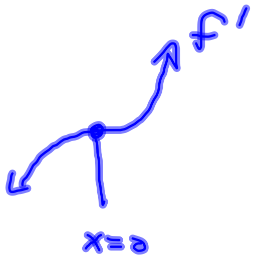
GCF

$4x^3$

$\Rightarrow 4x^2(x+6) = 0$

$\Rightarrow x = 0$ OR $x = -6$





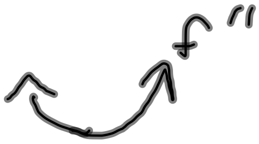
Dec: $(-\infty, -6)$

Inc: $(-6, \infty)$

Local min @ $x = -6$

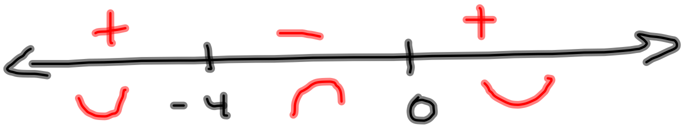
$f(-6) = -232 \rightsquigarrow \boxed{(-6, -232) \text{ MIN}}$

$f''(x) = 12x^2+48x \overset{\text{SET}}{=} 0$



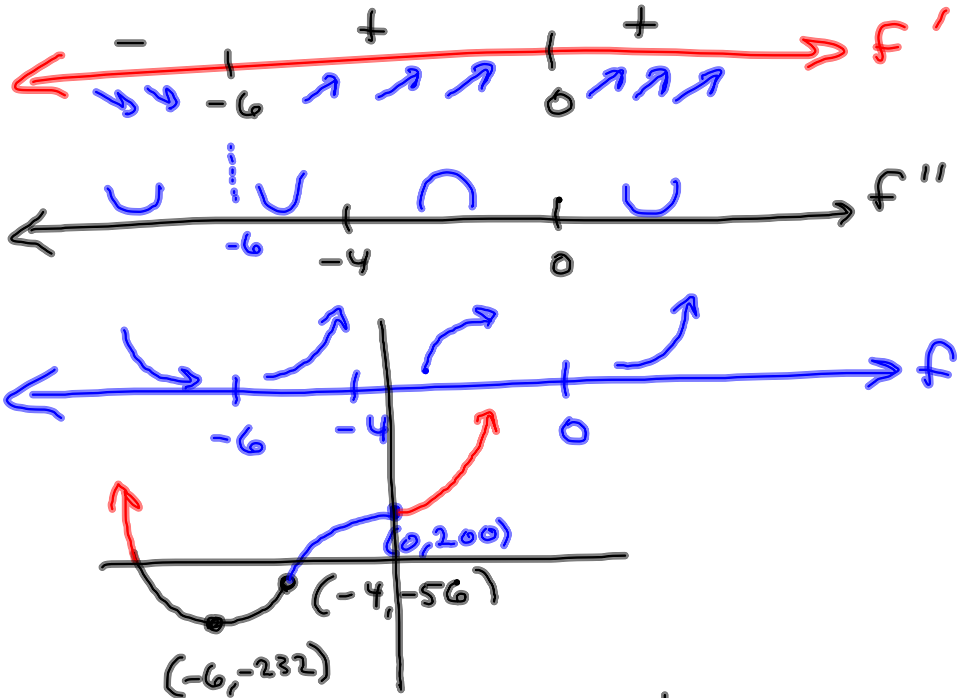
$\Rightarrow 12x(x+4)=0$

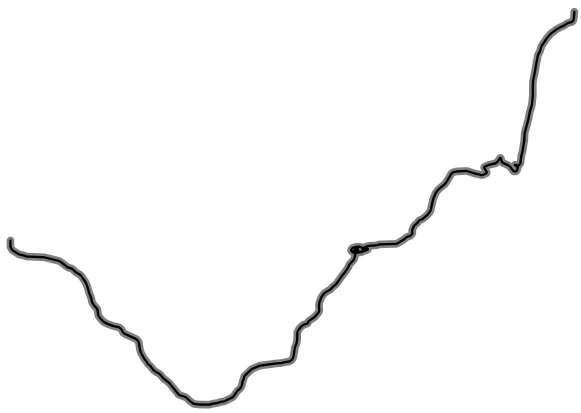
$x=-4,\ 0$



c-up: $(-\infty,-4)\cup(0,\infty)$

c-down: $(-4,0)$





#56 b

Given:
$f(x) > 0$ & $f''(x) > 0$

Claim:
$g(x) = [f(x)]^2$ is also concave up

i.e., $g''(x) > 0$

Proof

$$g'(x) = \overbrace{2f(x)f'(x)}^{\text{Chain Rule on } [f(x)]^2} \implies$$

$$g''(x) = \underbrace{2f'(x)f'(x) + 2f(x)f''(x)}_{\text{Product Rule}}$$

$$= 2\underbrace{[f'(x)]^2}_{\geq 0 \text{, since it's a square}} + 2\underbrace{f(x)f''(x)}_{>0 \text{, since } f(x)>0 \text{ \& } f''(x)>0} > 0 \quad \blacksquare$$

43–44
(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.

(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

43. $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$

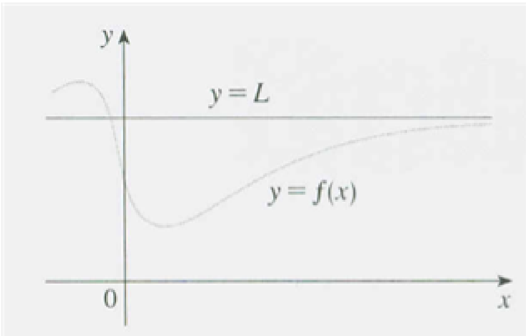
http://people.hofstra.edu/stefan_waner/realworld/functions/func.html

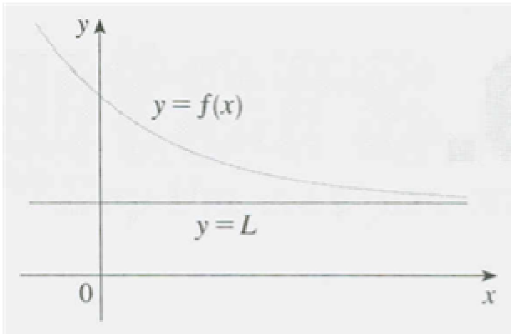
4.4 LIMITS AT INFINITY; HORIZONTAL ASYMPTOTES

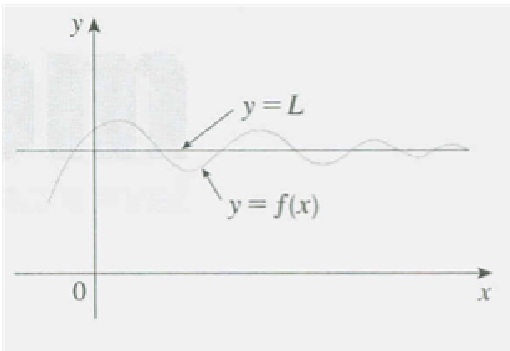
1 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

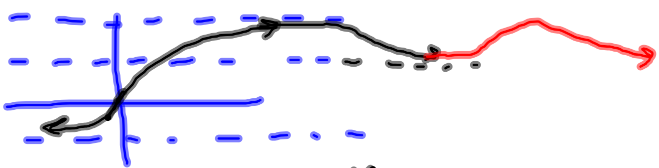
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.







How many H.A. can a function have? 2



$$\frac{x}{\sqrt{x^2+1}}$$

2 DEFINITION Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

3 DEFINITION The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L_1 \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L_2$$

$L_1 = 1$, $L_2 = -1$

Main examples of horizontal asymptotes: Rational functions where the degree of the numerator is the same as the degree of the denominator.

If degree of denominator is Greater than the degree of the numerator, then $R(x) = \frac{p(x)}{q(x)}$ is PROPER and $y = 0$ is the HA.

9–30 Find the limit.

10. $\lim_{x\to\infty} \frac{3x+5}{x-4}$ $= 3$

$\lim_{|x|\to\infty} \frac{a}{x} = 0$

$$\frac{3x+5}{x-4} = \frac{x\left(3+\frac{5}{x}\right)}{x\left(1-\frac{4}{x}\right)} = \frac{3+\frac{5}{x}}{1-\frac{4}{x}} \xrightarrow{x\to\infty} \frac{3}{1} = 3$$

$\frac{5}{x} \to 0$, $\frac{4}{x} \to 0$

13. $\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4}$

17. $\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$ $= \lim_{x\to\infty} \frac{\sqrt{9x^6\left(1-\frac{1}{9x^5}\right)}}{x^3\left(1+\frac{1}{x^3}\right)}$. Now,

$$\frac{\sqrt{9x^6-x}}{x^3+1} = \frac{3|x|^3\sqrt{1-\frac{1}{9x^5}}}{x^3\left(1+\frac{1}{x^3}\right)} = \frac{3x^3\sqrt{1-\frac{1}{9x^5}}}{x^3\left(1+\frac{1}{x^3}\right)}$$

(since $x \to \infty$ implies, in particular, that $x > 0$)

$$= \frac{3\sqrt{1-\frac{1}{9x^5}}}{1+\frac{1}{x^3}} \xrightarrow{x\to\infty} 3$$

The following is an "odd" application of the "rationalization technique."

$$(3x)^2 = 3^2x^2 = 9x^2$$

19. $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)$ $= \infty - \infty$!?

$$= \lim_{x\to\infty} \left[\left(\frac{\sqrt{9x^2+x} - 3x}{1}\right)\left(\frac{\sqrt{9x^2+x} + 3x}{\sqrt{9x^2+x} + 3x}\right)\right]$$

$$= \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2\left(1+\frac{1}{9x}\right)}+3x} = \lim_{x\to\infty} \frac{x}{|3x|\sqrt{1+\frac{1}{9x}}+3x}$$

$$= \lim_{x\to\infty} \frac{x}{3x\sqrt{1+\frac{1}{9x}}+3x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}}{3\cancel{x}\left(\sqrt{1+\frac{1}{9x}}+1\right)} = \lim_{x\to\infty} \frac{1}{3\left(\sqrt{1+\frac{1}{9x}}+1\right)}$$

$$= \frac{1}{3(\sqrt{1}+1)} = \frac{1}{3(1+1)} = \frac{1}{3(2)} = \frac{1}{6}$$

So, in a sense, $\sqrt{9x^2+x} \approx 3x + \frac{1}{6}$ as $x \to \infty$!?

$$\lim_{x\to\infty} (\cos x) \quad \nexists$$

$\boxed{4}$ **THEOREM** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Mar 6-10:04 AM

5 DEFINITION Let f be a function defined on some interval (a, ∞). Then

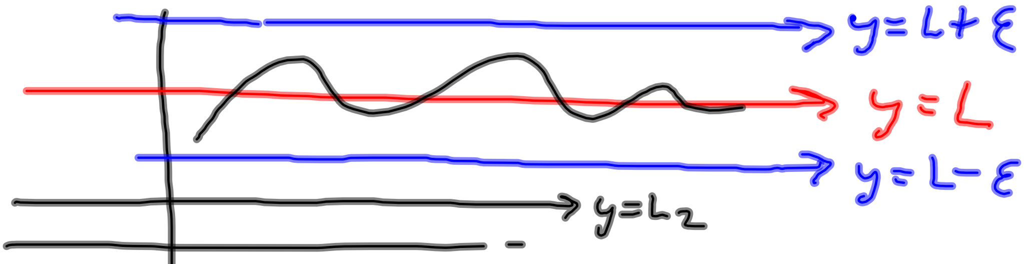
$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

We have a small number of exercises based on this formal definition, but we won't approach them in general; rather, we will be given a fixed epsilon, and then use a grapher to find an appropriate value of N.

PONDER



$x = N$ This is why you can't have more than one H.A.

$x \to \infty$ for $f(x)$ off to the right

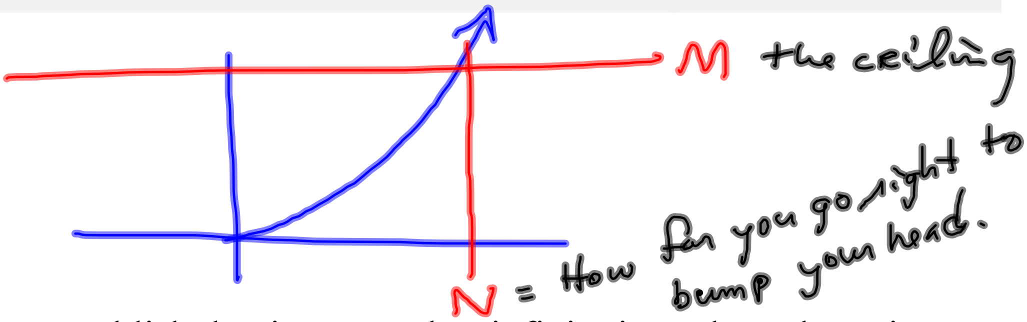
$x \to -\infty$ And no more than one off to the left

7 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$



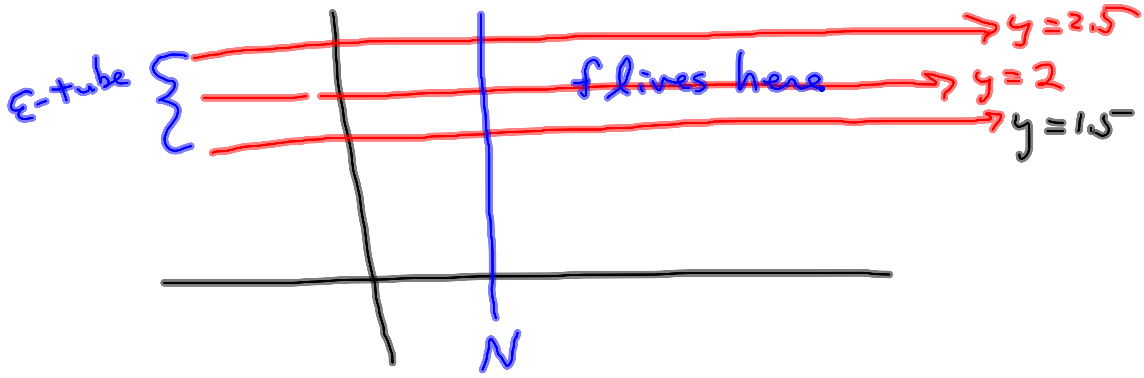
Basically, to establish that it approaches infinity is to show that, given any big number M, you can find a value of $x = N$ such that the function (y) stays above M whenever $x > N$.

"I can make it bigger than any fixed number. That means it's approaching infinity."

§4.4 #62 Pictures!

$$\lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{x+1} = 2$$

find N so that $f(x) = \frac{\sqrt{4x^2+1}}{x+1}$ is within $\varepsilon = .5$ of $L = 2$



$(4, f(4))$ must be a max.

$f'(4) = 0$ f(x) is horizontal
It's flat

$f''(4) = -3$ f(x) if frowning

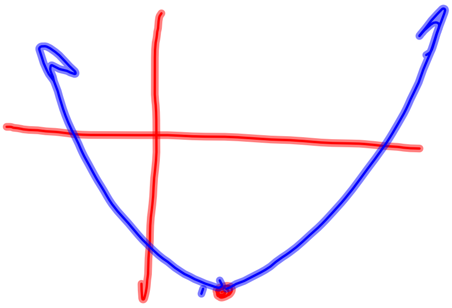
$f(x) = x^2 - 8x + 6$
$f'(x) = 2x - 8$
$f''(x) = 2$

$f'(x) = 0 \Rightarrow x = 4$ $f(x) = \frac{x^2}{2} + 4x + 3$

$f(4) = -10$
$(4, -10)$ on graph of f. This means $f'(4) = 0$
Max/Min candidate.

$x^2 - 8x + 6 = f(x)$

$f''(4) = 2 > 0$



The closest thing in practice
to $\lim\limits_{x\to\infty} f(x) = \infty$ is End Behavior:

$f(x) = 11x^4 - 3x^3 + 4x^2 - 571$
What power function does $f(x)$ resemble
as $|x| \longrightarrow \infty$?

$11x^4$

$-13x^5$ + smaller degree terms.

See #48.

§4.4 Get Crackin'
Due Friday.

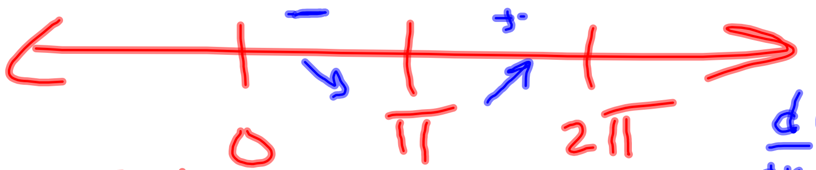
4.3 #39

$f'(\theta) = -2\sin\theta(1+\cos\theta) \overset{SET}{=} 0$

$f'(\theta) = 0$ @ $\Rightarrow -2\sin\theta = 0$ OR $1+\cos\theta = 0$

$\sin\theta = 0$ OR $\cos\theta = -1$

$0, \pi, 2\pi$ $\qquad \pi$



Test

$(0,\pi)$ $\quad \frac{\pi}{2}$ $\quad f'(\frac{\pi}{2}) = -2 < 0$

$(\pi,2\pi)$ $\quad \frac{3\pi}{2}$ $\quad f'(\frac{3\pi}{2}) = +2 > 0$

$\frac{d}{dx}\left[(x^2+2)^5\right] =$

$5(x^2+2)^4(2x)$

$= \frac{d}{dx}\left[(\cos x)^2\right]$

$= 2(\cos x)^1(-\sin x)$

$2f(x)f'(x)$